Supplementary Material for Cramér–Rao Bounds for Filtering Based on Gaussian Process State-Space Models

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Supplementary Material for "Cramér-Rao Bounds for Filtering Based on Gaussian Process State-Space Models"

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Abstract

This report contains supplementary material for the paper [1].

1 Derivations of auxiliary Fisher information matrix

In this section, the detailed derivations for computing the auxiliary Fisher information submatrix in [1] is provided. Similar proof can also be found in [2]. Considering the system models given in [1, Fig. 2(b)], and the likelihood distribution as given in [1, Section III.B], the distribution of $p(\boldsymbol{y}_{0:k}, \boldsymbol{z}_{1:k}|\boldsymbol{x}_{0:k})$ can be partitioned as

$$\begin{aligned} p(\boldsymbol{y}_{0:k}, \boldsymbol{z}_{1:k}|\boldsymbol{x}_{0:k}) &= p(\boldsymbol{y}_{0:k}|\boldsymbol{x}_{0:k}, \mathcal{D})p(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{0:k}) \\ &= p(\boldsymbol{y}_k|\boldsymbol{x}_k, \mathcal{D})p(\boldsymbol{y}_{0:k-1}|\boldsymbol{x}_{0:k-1}, \mathcal{D}) \\ &\times p(\boldsymbol{z}_k|\boldsymbol{x}_{0:k})p(\boldsymbol{z}_{1:k-1}|\boldsymbol{x}_{0:k-1}) \\ &= p(\boldsymbol{y}_k|\boldsymbol{x}_k, \mathcal{D})p(\boldsymbol{z}_{1:k-1}|\boldsymbol{x}_{0:k-1}) \\ &\times p(\boldsymbol{y}_{0:k-1}, \boldsymbol{z}_{1:k-1}|\boldsymbol{x}_{0:k-1}). \end{aligned} \tag{1}$$

From the definition of the auxiliary FIM, we have at time $k-1$

$$\begin{aligned} &J_{0:k-1}(\boldsymbol{x}_{0:k-1}) \\ &= -\mathbb{E}_{\boldsymbol{y}_{0:k-1}, \boldsymbol{z}_{1:k-1}}\left\{\Delta^{\boldsymbol{x}_{0:k-1}}_{\boldsymbol{x}_{0:k-1}} \ln p(\boldsymbol{y}_{0:k-1}, \boldsymbol{z}_{1:k-1}|\boldsymbol{x}_{0:k-1})|\boldsymbol{x}_{0:k-1}\right\} = \\ &-\mathbb{E}_{\boldsymbol{y}_{0:k-1}, \boldsymbol{z}_{1:k-1}}\left\{\begin{bmatrix} \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}} \\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} \end{bmatrix} \ln p(\boldsymbol{y}_{0:k-1}, \boldsymbol{z}_{1:k-1}|\boldsymbol{x}_{0:k-1})|\boldsymbol{x}_{0:k-1}\right\} \\ &= \begin{bmatrix} A_{k-1} & B_{k-1} \\ B^T_{k-1} & C_{k-1}. \end{bmatrix} \end{aligned} \tag{2}$$

$$
\begin{aligned}
&J_{0:k}(\boldsymbol{x}_{0:k}) = \\
&- \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ \begin{bmatrix} \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{0:k-2}} \\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k-1}} \\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k}} \end{bmatrix} \ln p(\boldsymbol{y}_k|\boldsymbol{x}_k, \mathcal{D})|\boldsymbol{x}_{0:k} \right\} \\
&- \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ \begin{bmatrix} \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{0:k-2}} \\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k-1}} \\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k}} \end{bmatrix} \ln p(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{x}_{k-1})|\boldsymbol{x}_{0:k} \right\} \\
&- \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ \begin{bmatrix} \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{0:k-2}} \\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k-1}} \\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k}} \end{bmatrix} \ln p(\boldsymbol{y}_{0:k-1}, \boldsymbol{z}_{1:k-1}|\boldsymbol{x}_{0:k-1}, \mathcal{D})|\boldsymbol{x}_{0:k-1} \right\} \\
&= -\mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k}} \end{bmatrix} \ln p(\boldsymbol{y}_k|\boldsymbol{x}_k, \mathcal{D})|\boldsymbol{x}_{0:k} \right\} \\
&- \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k-1}} \\ 0 & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k}} & \Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k}} \end{bmatrix} \ln p(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{x}_{k-1})|\boldsymbol{x}_{0:k} \right\} \\
&+ \begin{bmatrix} A_{k-1} & B_{k-1} & 0 \\ B^T_{k-1} & C_{k-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A_{k-1} & B_{k-1} & 0 \\ B^T_{k-1} & C_{k-1} + D^{11}_k & D^{12}_k \\ 0 & (D^{12}_k)^T & D^{22}_k \end{bmatrix}, \qquad (5)
\end{aligned}
$$

From (2), the auxiliary Fisher information submatrix $J_{k-1}(\boldsymbol{x}_{0:k})$ can be computed as the inverse of the $p \times p$ lower-right partition of $J^{-1}_{0:k-1}(\boldsymbol{x}_{0:k})$, and using block matrix inversion, the following is obtained

$$
J_{k-1}(\boldsymbol{x}_{0:k}) = C_{k-1} - B^T_{k-1} A^{-1}_{k-1} B_{k-1}. \qquad (3)
$$

At time k, by substituting (1) into the definition of the auxiliary FIM, we can derive (5), where D^{11}_k, D^{12}_k and D^{22}_k are give as

$$
\begin{aligned}
D^{11}_k &= \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ -\Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} \ln p(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{x}_{k-1})|\boldsymbol{x}_{0:k} \right\} \qquad (4a) \\
D^{12}_k &= \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ -\Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k-1}} \ln p(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{x}_{k-1})|\boldsymbol{x}_{0:k} \right\} \qquad (4b) \\
D^{22}_k &= \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ -\Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k}} \ln p(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{x}_{k-1})|\boldsymbol{x}_{0:k} \right\} \\
&\quad + \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{z}_{1:k}} \left\{ -\Delta^{\boldsymbol{x}_{k}}_{\boldsymbol{x}_{k}} \ln p(\boldsymbol{y}_k|\boldsymbol{x}_k)|\boldsymbol{x}_{0:k} \right\}. \qquad (4c)
\end{aligned}
$$

The auxiliary Fisher information submatrix $J_k(\boldsymbol{x}_{0:k})$ is computed as the inverse of the $p \times p$ lower-right partition of $J^{-1}_{0:k}(\boldsymbol{x}_{0:k})$. With (5), the block matrix inversion yields

$$
\begin{aligned}
&J_k(\boldsymbol{x}_{0:k}) = \\
&D^{22}_k - \left(D^{12}_k\right)^T \left[C_{k-1} + D^{11}_k - B^T_{k-1} A^{-1}_{k-1} B_{k-1}\right]^{-1} D^{12}_k \qquad (6)
\end{aligned}
$$

By substituting (3) into (6), we obtain the recursive expression for $J_k(\boldsymbol{x}_{0:k})$

$$J_k(\boldsymbol{x}_{0:k}) = D_k^{22} - \left(D_k^{12}\right)^T \left[D_k^{11} + J_{k-1}(\boldsymbol{x}_{0:k-1})\right]^{-1} D_k^{12}. \tag{7}$$

2 Derivations of Bayesian information matrix

In this section, the detailed derivation for computing the BIM iteratively is provided. From the definition of BIM, we have at time $k-1$

$$\begin{aligned}
&J_{\mathrm{B},0:k-1} = \mathbb{E}_{\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1}}\left\{-\Delta^{\boldsymbol{x}_{0:k-1}}_{\boldsymbol{x}_{0:k-1}} \ln p(\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1})\right\}\\
&= -\mathbb{E}_{\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1}}\left\{\Delta^{\boldsymbol{x}_{0:k-1}}_{\boldsymbol{x}_{0:k-1}} \ln p(\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1})\right\}\\
&= -\mathbb{E}_{\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1}}\left\{\begin{bmatrix}\Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}}\\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}}\end{bmatrix} \ln p(\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1})\right\}\\
&= \begin{bmatrix}A_{\mathrm{B},k-1} & B_{\mathrm{B},k-1}\\ B^T_{\mathrm{B},k-1} & C_{\mathrm{B},k-1}.\end{bmatrix}
\end{aligned} \tag{8}$$

With block matrix inversion, the Fisher information submatrix at time $k-1$, $J_{\mathrm{B},k-1}$, is computed as the inverse of the $p \times p$ lower-right partition of $J^{-1}_{\mathrm{B},0:k-1}$

$$J_{\mathrm{B},k-1} = C_{\mathrm{B},k-1} - B^T_{\mathrm{B},k-1} A^{-1}_{\mathrm{B},k-1} B_{\mathrm{B},k-1}. \tag{9}$$

At time k, the joint distribution of $\boldsymbol{y}_{0:k}$ and $\boldsymbol{x}_{0:k}$, for the state-space models shown in [1, Fig. 2], can be partitioned as

$$\begin{aligned}
&p(\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}) = p(\boldsymbol{y}_{0:k}|\boldsymbol{x}_{0:k})p(\boldsymbol{x}_{0:k})\\
&= p(\boldsymbol{y}_k|\boldsymbol{x}_k)p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})p(\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1}).
\end{aligned} \tag{10}$$

The BIM at time k can be computed by substituting (10)

$$
\begin{aligned}
&J_{\mathrm{B},0:k} = \\
&-\mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}}\left\{\begin{bmatrix}\Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{0:k-2}}\\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{k-1}}\\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_k} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_k} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_k}\end{bmatrix}\ln p(\boldsymbol{y}_k|\boldsymbol{x}_k)\right\}\\
&-\mathbb{E}_{\boldsymbol{y}_{0k},\boldsymbol{x}_{0k}}\left\{\begin{bmatrix}\Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{0:k-2}}\\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{k-1}}\\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_k} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_k} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_k}\end{bmatrix}\ln p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})\right\}\\
&-\mathbb{E}_{\boldsymbol{y}_{0k},\boldsymbol{x}_{0k}}\left\{\begin{bmatrix}\Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{0:k-2}} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{0:k-2}}\\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{k-1}}\\ \Delta^{\boldsymbol{x}_{0:k-2}}_{\boldsymbol{x}_k} & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_k} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_k}\end{bmatrix}\ln p(\boldsymbol{y}_{0:k-1},\boldsymbol{x}_{0:k-1})\right\}\\
&= -\mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}}\left\{\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_k}\end{bmatrix}\ln p(\boldsymbol{y}_k|\boldsymbol{x}_k)\right\}\\
&-\mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}}\left\{\begin{bmatrix}0 & 0 & 0\\ 0 & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{k-1}}\\ 0 & \Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_k} & \Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_k}\end{bmatrix}\ln p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})\right\}\\
&+\begin{bmatrix}A_{\mathrm{B},k-1} & B_{\mathrm{B},k-1} & 0\\ B^T_{\mathrm{B},k-1} & C_{\mathrm{B},k-1} & 0\\ 0 & 0 & 0\end{bmatrix}\\
&= \begin{bmatrix}A_{\mathrm{B},k-1} & B_{\mathrm{B},k-1} & 0\\ B^T_{\mathrm{B},k-1} & C_{\mathrm{B},k-1}+L^{11}_k & L^{12}_k\\ 0 & (L^{12}_k)^T & L^{22}_k\end{bmatrix},
\end{aligned}
\tag{11}
$$

where L^{11}_k, L^{12}_k and L^{22}_k are defined as

$$L^{11}_k = \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}}\left\{-\Delta^{\boldsymbol{x}_{k-1}}_{\boldsymbol{x}_{k-1}}\ln p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})\right\} \tag{12a}$$

$$L^{12}_k = \left(L^{21}_k\right)^T = \mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}}\left\{-\Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_{k-1}}\ln p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})\right\} \tag{12b}$$

$$\begin{aligned}L^{22}_k = &\mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}}\left\{-\Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_k}\ln p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})\right\}\\ &+\mathbb{E}_{\boldsymbol{y}_{0:k},\boldsymbol{x}_{0:k}}\left\{-\Delta^{\boldsymbol{x}_k}_{\boldsymbol{x}_k}\ln p(\boldsymbol{y}_k|\boldsymbol{x}_k)\right\}.\end{aligned} \tag{12c}$$

Hence, using block matrix inversion, the Bayesian information submatrix $J_{\mathrm{B},k}$ is computed as the inverse of the $p \times p$ lower-right partition of $J^{-1}_{\mathrm{B},0:k}$ yielding

$$
\begin{aligned}
J_{\mathrm{B},k} &= L^{22}_k - \left(L^{12}_k\right)^T\left[L^{11}_k + C_{\mathrm{B},k-1} - B^T_{\mathrm{B},k-1}A^{-1}_{\mathrm{B},k-1}B_{\mathrm{B},k-1}\right]^{-1}L^{12}_k\\
&= L^{22}_k - \left(L^{12}_k\right)^T\left[L^{11}_k + J_{\mathrm{B},k-1}\right]^{-1}L^{12}_k.
\end{aligned}
\tag{13}
$$

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