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***Abstract*—In this paper we propose a method to exactly certify the complexity of an active-set method which is based on reformulating strictly convex quadratic programs to nonnegative least-squares problems. The exact complexity of the method is determined by proving the correspondence between the method and a standard primal active-set method for quadratic programming applied to the dual of the quadratic program to be solved. Once this correspondence has been established, a complexity certification method which has already been established for the primal active-set method is used to also certify the complexity of the nonnegative least-squares method. The usefulness of the proposed method is illustrated on a multi-parametric quadratic program originating from model predictive control of an inverted pendulum.**

***Index Terms*—Optimization algorithms, Predictive control for linear systems**

I. Introduction

An optimization problem has to be solved in each time-instant when model predictive control (MPC) is used for control. The optimization problems in question are often quadratic programs (QPs) and to be able to use MPC in embedded systems, the employed QP solvers need to be simple, fast and have real-time guarantees. Popular methods for solving QPs originating from MPC are interior-point methods [1][2], gradient projection methods [3][4][5], the alternating method of multipliers (ADMM) [6] and active-set methods [7][8][9][10][11]. The active-set method in [11], which is based on reformulating strict convex quadratic programs to nonnegative least squares (NNLS) problems, is simple to code and has been shown to be efficient for solving QPs originating from MPC. However, since it is an active-set method, its complexity can be exponential in the worst case.

To be able to provide tight real-time guarantees for active-set methods, complexity certification methods which determine the worst-case behaviour for the active-set methods in [7],[8] and [10] have been presented in [12],[13] and [14], respectively. For a given MPC problem, these methods determine *exactly* which subproblems, i.e., systems of linear equations, that have to be solved to find the solution, for every possible QP that needs to be solved. A unifying complexity certification framework for a class of standard active-set methods, which covers both the methods from [12] and [13], is available in [15].

In this paper we extend the possibility to also certify the complexity of the efficient QP method presented in [11], adding to its simplicity and efficiency the possibility to determine its exact complexity, increasing its practical applicability. This is done by proving that the working-set changes of the QP method are equivalent to a standard primal active-set method applied to the dual of the QP to be solved. This equivalence allows direct use of the complexity certification framework in [15]. The main focus of this paper is, hence, not to devise another complexity certification method from scratch, but to relate the method presented in [11] to the active-set method considered in [15], for which there exists a complexity certification method. In summary, the main contribution of this paper is proving the equivalence between the QP methods in [11] and [15], and from this equivalence the technical contribution of a method which certifies the exact complexity of the QP method in [11] follows.

II. Problem formulation

Consider a multi-parametric quadratic program (mpQP) in the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2}x^T H x + (f^T + \theta^T f_\theta^T)x \\ \text{subject to} \quad & Ax \leq b + W\theta, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and the parameter $\theta \in \Theta_0 \subseteq \mathbb{R}^p$, with Θ_0 being a polytope (such as a box). The mpQP is given by $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $W \in \mathbb{R}^{m\times p}$, $f \in \mathbb{R}^n$, $f_\theta \in \mathbb{R}^{n\times p}$, and $H \in \mathbb{S}^n_{++}$. The minimizer of (1), given θ, is denoted $x^*(\theta)$. It is well-known that a linear MPC problem can be cast in the form (1), where the parameter θ contains the measured/estimated states and reference signals [16].

In [11] it is shown that by introducing

$$M \triangleq A\mathcal{L}^{-1}, \quad d(\theta) \triangleq b + W\theta + AH^{-1}(f + f_\theta\theta), \tag{2}$$

where $\mathcal{L}$ is the Cholesky factor of H, (1) can be restated as the nonnegative least-squares problem

$$\underset{y\geq 0}{\text{minimize}} \quad \frac{1}{2}\left\| \begin{bmatrix} M^T \\ d(\theta)^T \end{bmatrix} y + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \right\|_2^2 \tag{3}$$

where γ is any positive scalar, $M \in \mathbb{R}^{m\times n}$, $d(\theta) \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$. Furthermore, the relationship between $x^*(\theta)$ and the minimizer of (3), $y^*(\theta)$, is

$$x^*(\theta) = -H^{-1}\Big(f + f_\theta\theta + \frac{1}{\gamma + d(\theta)^T y^*(\theta)} A^T y^*(\theta)\Big) \tag{4}$$

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Remark 1: It is straightforward to extend the ideas in this paper to also handle equality constraints in (1) by using the results in [17]. However, for the sake of a clean presentation, we will only consider inequality constraints.

A. Notation

Since the algorithm to be studied is iterative, we use a subscript k to denote the value at iteration k for quantities that change between iterations. E.g., y_k denotes the value of y at iteration k. Of importance is also the so-called working set $\mathcal{W}_k$ which contains a subset of the components of y_k that are free to vary. Likewise, the set of components that are not in $\mathcal{W}_k$ is denoted $\bar{\mathcal{W}}_k$ and contains components that are fixed at zero. For indexing of matrices, $[N]_i$ denotes the ith row of matrix N and $[N]_{\mathcal{W}_k}$ denotes the submatrix obtained by extracting the rows of N indexed by $\mathcal{W}_k$. The shorter notation $N_k \triangleq [N]_{\mathcal{W}_k}$ for matrices that do not change between iterations is also introduced for convenience.

B. Nonnegative least-squares method

The nonnegative least-squares method for solving (1) presented in [11] is described briefly below and summarized in Algorithm 1. For a more detailed description, see [11] or [18, Sec. 23.3]. The main objective of Algorithm 1 is to retain nonnegativity of the iterate y while updating the working set $\mathcal{W}$. At iteration k, the least-squares (LS) problem

$$\begin{array}{ll} \underset{y}{\text{minimize}} & \frac{1}{2}\left\| \begin{bmatrix} M^T \\ d(\theta)^T \end{bmatrix} y + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \right\|_2^2 \\ \text{subject to} & [y]_i = 0, \quad i \in \bar{\mathcal{W}}_k \end{array} \tag{5}$$

defined by the current working set $\mathcal{W}_k$ is solved, with the solution of (5) being denoted y_k^*. The iterate y_k is then updated to y_{k+1} by a line search from y_k to y_k^*. To retain nonnegativity, the first component of y_k which becomes negative during this line search is removed from $\mathcal{W}_k$. If no such component exists, i.e., if $y_k^* \geq 0$, global optimality is checked for y_k^* by investigating the dual variable w_k. If w_k is nonnegative, a global optimum has been found, otherwise, the index of the most negative component of w_k is added to $\mathcal{W}$. When the working set has been updated, another LS problem defined by the new working set is solved and the steps above are repeated until global optimality is reached.

The choice of γ is not critical since any $\gamma > 0$ is sufficient for the algorithm to work. In [18, Sec. 23] $\gamma = 1$ is used, and in [11] γ is adaptively updated by adding or removing the absolute value of $[d]_i$ when i is added or removed from $\mathcal{W}$, respectively. In this paper we consider γ to be constant for simplicity. However, the results also extends to an adaptively changing γ since the working-set changes produced by Algorithm 1 are independent of γ.

Remark 2: The presentation of Algorithm 1 is slightly modified compared with [11] to make the definition of an iteration in the algorithm clearer. Furthermore, some checks for infeasibility that are included in [11] have been omitted to clean up the algorithm, i.e., we assume that (1) is primal feasible for all θ of interest. This condition can be immediately verified, for example, by checking that QP (1) is feasible for all vertices θ_i of Θ_0, as is shown in Lemma 1.

Lemma 1: Let $\{\theta_i\}_{i=1}^M$ be the vertices of the polytope Θ_0 and let $X_i \triangleq \{x \in \mathbb{R}^n : Ax \leq b + W\theta_i\}$ be the feasible set for (1) when $\theta = \theta_i$. Then problem (1) is feasible $\forall \theta \in \Theta_0 \Leftrightarrow X_i \neq \emptyset$, $\forall i \in \{1, \ldots, M\}$.

Proof: Since $\theta_i \in \Theta_0$, $\forall i$, the left-to-right implication follows immediately. For the right-to-left implication we have, since Θ_0 is convex, $\theta = \sum_{i=1}^M \alpha_i \theta_i$, $\sum \alpha_i = 1$, $\alpha_i \geq 0$. Let $x_i \in X_i$ and consider $x = \sum_{i=1}^M \alpha_i x_i$. Then $Ax = \sum \alpha_i A x_i \leq \sum \alpha_i (b + W\theta_i) = b + W \sum \alpha_i \theta_i = b + W\theta$. ∎

Algorithm 1 Given θ, solve the mpQP (1) with NNLS [11]

1: $v \leftarrow \mathcal{L}^{-T}(f_\theta \theta + f)$; $\quad d \leftarrow (b + W\theta) + Mv$
2: $k \leftarrow 1$; $\mathcal{W}_k \leftarrow \emptyset$; $y_k \leftarrow 0$
3: **while** true **do**
4: $\quad y_k^* \leftarrow$ solution to least squares problem (5)
5: $\quad$ **if** $y_k^* \geq 0$ **then**
6: $\quad\quad w_k \leftarrow M(M_k^T [y_k^*]_{\mathcal{W}_k}) + (\gamma + d_k^T [y_k^*]_{\mathcal{W}_k}) d$
7: $\quad\quad$ **if** $w_k \geq -(\gamma + d_k^T [y_k^*]_{\mathcal{W}_k})\epsilon$ **then**
8: $\quad\quad\quad$ **go to** step 15
9: $\quad\quad$ **else** $i \leftarrow \underset{i \in \bar{\mathcal{W}}_k}{\arg\min}\, [w_k]_i$, $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \cup \{i\}$
10: $\quad\quad\quad y_{k+1} \leftarrow y_k^*$
11: $\quad$ **else** $l \leftarrow \underset{h \in \mathcal{W}: [y_k^*]_h < 0}{\arg\min} \{\frac{[y_k]_h}{[y_k]_h - [y_k^*]_h}\}$
12: $\quad\quad \alpha_k \leftarrow \frac{[y]_l}{[y]_l - [y_k^*]_l}$; $\quad y_{k+1} \leftarrow y_k + \alpha_k (y_k^* - y_k)$
13: $\quad\quad \mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \setminus \{l\}$
14: $\quad k \leftarrow k + 1$
15: **return** $\lambda^* \leftarrow \frac{y_k^*}{\gamma + d_k^T y_k^*}$, $x^* \leftarrow -\mathcal{L}^{-1}(M_k^T \lambda_k^* + v)$, $\mathcal{W}_k$

III. Properties of NNLS algorithm

This section describes properties of Algorithm 1 that will be central in Section IV, where a complexity certification method for Algorithm 1 is outlined.

Analyzing the parametric behaviour for Algorithm 1 does not immediately follow from previous work on the topic since there is parameter dependence in the quadratic term in the objective function of (3) due to $d(\theta)$. All of the complexity certification methods in [13],[14] and [15] rely on parameter dependence only appearing in linear terms. Hence, the main objective in this paper will be to disentangle the parameter dependence in the quadratic term and to show that Algorithm 1 will be equivalent, in terms of working-set changes, to another algorithm which operates on a problem with only parameter dependence in the linear term. This disentanglement is done by considering λ which is a linear fractional transformation of y defined as

$$\lambda \triangleq \frac{y}{\gamma + d^T y}. \tag{6}$$

The following scalars will also prove useful

$$\sigma_k \triangleq \gamma + d^T y_k^*, \quad \rho_k \triangleq \gamma + d^T y_k. \tag{7}$$

With these scalars we have $y_k = \rho_k \lambda_k$ and $y_k^* = \sigma_k \lambda_k^*$ if $\rho_k \neq 0$ and $\sigma_k \neq 0$, respectively.

For clarity, we initially deduce properties of Algorithm 1 under the assumption that $M_k M_k^T$ is nonsingular. In Section III-D we discuss properties of the algorithm in the singular case.

A. Least-squares subproblems

The solution y_k^* to the subproblem (5) can be found by solving the following KKT-system

$$\begin{pmatrix} MM^T + dd^T & [I]_{\bar{\mathcal{W}}_k}^T \\ [I]_{\bar{\mathcal{W}}_k} & 0 \end{pmatrix} \begin{pmatrix} y_k^* \\ w_k \end{pmatrix} = \begin{pmatrix} -\gamma d \\ 0 \end{pmatrix}, \tag{8}$$

where I is the $m \times m$ identity matrix. (8) has the solution

$$y_k^* = -\gamma I_k^T (M_k M_k^T + d_k d_k^T)^{-1} d_k. \tag{9}$$

Recall from Section II-A that M_k, d_k and I_k is shorthand notation for submatrices obtained when indexing with $\mathcal{W}_k$, i.e., $[M]_{\mathcal{W}_k}$, $[d]_{\mathcal{W}_k}$ and $[\mathcal{I}]_{\mathcal{W}_k}$, respectively. Another way of finding the solution to (5) is to directly set all components of y_k^* that are not in $\mathcal{W}_k$ to zero, resulting in an unconstrained optimization problem which is solved by the linear system

$$(M_k M_k^T + d_k d_k^T)[y_k^*]_{\mathcal{W}_k} = -\gamma d_k, \tag{10}$$

This can be rewritten as

$$M_k M_k^T [y_k^*]_{\mathcal{W}_k} = -d_k(\gamma + d_k^T [y_k^*]_{\mathcal{W}_k}) = -\sigma_k d_k, \tag{11}$$

where the last equality follows from $[y_k^*]_{\bar{\mathcal{W}}_k} = 0$ which gives

$$\gamma + d_k^T [y_k^*]_{\mathcal{W}_k} = \gamma + d^T y_k^* = \sigma_k. \tag{12}$$

Our goal is now to formulate a corresponding KKT-system for $\lambda_k^* \triangleq \frac{y_k^*}{\sigma_k}$ and $\mu_k \triangleq \frac{w}{\sigma_k}$ instead of y_k^* and w_k, respectively. First, to ensure λ_k^* and μ_k are well-defined, we ensure that division of σ_k is valid, i.e., that σ_k is nonzero when $M_k M_k^T$ is nonsingular. Even more strongly, σ_k is positive, as proved by the following lemma.

Lemma 2: $\sigma_k \triangleq \gamma + d^T y_k^* > 0$, if $M_k M_k^T$ is nonsingular.

Proof: When $M_k M_k^T$ is nonsingular, (11) gives that

$$[y_k^*]_{\mathcal{W}_k} = -\sigma_k (M_k M_k^T)^{-1} d_k. \tag{13}$$

Inserting this into the definition of σ_k gives

$$\begin{aligned} &\sigma_k = \gamma + d_k^T [y_k^*]_{\mathcal{W}_k} = \gamma - \sigma_k d_k^T (M_k M_k^T)^{-1} d_k \Leftrightarrow \\ &\sigma_k = \gamma / (1 + d_k^T (M_k M_k^T)^{-1} d_k). \end{aligned} \tag{14}$$

By definition, $\gamma > 0$ and the denominator is nonnegative since $M_k M_k^T$ is positive definite, resulting in $\sigma_k > 0$. ∎

Now a KKT-system in terms of λ_k^* and μ_k, instead of y_k^* and w_k, can be formed by subtracting $dd^T y_k^*$ from both sides of the first equation of (8) and dividing both sides with $\sigma_k \neq 0$, resulting in

$$\begin{pmatrix} MM^T & I_{\bar{\mathcal{W}}_k}^T \\ I_{\bar{\mathcal{W}}_k} & 0 \end{pmatrix} \begin{pmatrix} \lambda_k^* \\ \mu_k \end{pmatrix} = \begin{pmatrix} -d \\ 0 \end{pmatrix}. \tag{15}$$

The solution to the KKT-system in (15) is

$$[\lambda_k^*]_{\mathcal{W}_k} = -(M_k M_k^T)^{-1} d_k, \tag{16a}$$

$$[\mu_k]_{\bar{\mathcal{W}}_k} = [M]_{\bar{\mathcal{W}}_k} M_k^T [\lambda_k^*]_{\mathcal{W}_k} + [d]_{\bar{\mathcal{W}}_k}, \tag{16b}$$

and $[\lambda_k^*]_{\bar{\mathcal{W}}_k} = 0$, $[\mu_k]_{\mathcal{W}_k} = 0$.

B. Checking for global optimality and adding index to $\mathcal{W}$

In Algorithm 1 the global optimum has been found if

$$w_k \geq -(\gamma + d^T y_k^*)\epsilon. \tag{17}$$

Otherwise, an index corresponding to the most negative component of w_k is added to $\mathcal{W}$, according to Line 9 of Algorithm 1.

The following lemma shows that the dual variable of the KKT-system in (15), μ_k, can be considered instead of w_k when checking for global optimality and for deciding which index that should be added to $\mathcal{W}$.

Lemma 3: When $M_k M_k^T$ is nonsingular

1) $w_k \geq -(\gamma + d^T y_k^*)\epsilon \Leftrightarrow \mu_k \geq -\epsilon$.
2) $\underset{j \in \bar{\mathcal{W}}_k}{\operatorname{argmin}} [w_k]_j = \underset{j \in \bar{\mathcal{W}}_k}{\operatorname{argmin}} [\mu_k]_j$.

Proof: Since $M_k M_k^T$ is nonsingular, $\sigma_k > 0$ from Lemma 2. Dividing both sides of $w_k \geq -(\gamma + d^T y_k^*)\epsilon$ with σ_k proves 1). Furthermore, the positiveness of σ_k gives argmin $[w_k]_j =$ argmin $[w_k]_j / \sigma_k =$ argmin $[\mu_k]_j$. ∎

C. Updating iterate and removing component from $\mathcal{W}$

The iterate y_k is updated in Line 12 of Algorithm 1 according to the line search

$$y_{k+1} = y_k + \alpha_k (y_k^* - y_k), \tag{18}$$

where $\alpha_k = \min_{h \in \mathcal{W}_k : [y_k^*]_h < 0} \alpha_k^h$, with α_k^i being defined as

$$\alpha_k^i \triangleq [y_k]_i / ([y_k - y_k^*]_i). \tag{19}$$

α_k^i can be interpreted as the step length taken from y_k in the direction $y_k - y_k^*$ which makes the i:th component of y zero. Also, note that $[y_k^*]_h < 0 \implies \alpha_k^h \in [0, 1)$.

Now, we are interested in the corresponding update of λ when y is updated according to (18). Inserting (18) in the definition of λ in (6) gives

$$\lambda_{k+1} = \frac{y_{k+1}}{\gamma + d^T y_{k+1}} = \frac{y_k + \alpha_k (y_k^* - y_k)}{\gamma + d^T (y_k + \alpha_k (y_k^* - y_k))}. \tag{20}$$

We will now show that the update of λ_k also can be seen as a line search.

Lemma 4: If $M_k M_k^T$ is nonsingular, $\exists \beta_k \in \mathbb{R}$ such that

$$\lambda_{k+1} = \lambda_k + \beta_k (\lambda_k^* - \lambda_k). \tag{21}$$

Proof: The lemma follows from linear fractional transformations conserving convex sets cf., e.g., [19, Sec. 2.3.3]. Concretely, picking

$$\beta_k = \frac{\alpha_k \sigma_k}{\alpha_k \sigma_k + (1 - \alpha_k)\rho_k} \tag{22}$$

and inserting it into (21) results in (20) by using (7). ∎

Furthermore, β_k^i is defined by (22) when α_k^i is used instead of α_k. Before considering properties of β_k^i, we prove that, similar to σ_k, $\rho_k > 0$ when $M_k M_k^T$ is nonsingular.

Lemma 5: If $M_k M_k^T$ is nonsingular, $\rho_k \triangleq \gamma + d^T y_k > 0$

Proof: The lemma is proven by induction. First, inserting y_{k+1} from (18) into the definition of ρ in (7) gives

$$\begin{aligned} \rho_{k+1} &= \gamma + d^T (y_k + \alpha_k (y_k^* - y_k)) \\ &= \gamma + d^T y_k + \alpha_k (d^T y_k^* - d^T y_k + \gamma - \gamma) \\ &= (1 - \alpha_k)\rho_k + \alpha_k \sigma_k. \end{aligned} \tag{23}$$

Now, assume that $\rho_k > 0$. Then $\rho_{k+1} > 0$ since $\alpha_k \in [0,1]$ and $\sigma_k > 0$ from Lemma 2. The base case is satisfied since $y_1 = 0 \implies \rho_1 = \gamma > 0$. Hence, the lemma follows by induction. ∎

This nonnegativity property of ρ_k, together with the nonnegativity property of σ_k, can be used to prove the following lemma which establishes a relation between α_k^i and β_k^i.

Lemma 6: If $M_k M_k^T$ is nonsingular and $\alpha_k^i, \alpha_k^j \in [0,1]$,

$$\alpha_k^i \leq \alpha_k^j \Leftrightarrow \beta_k^i \leq \beta_k^j. \tag{24}$$

Proof: Directly using the definition of β_k^i from (22), and dropping the subscript k for convenience, gives

$$\begin{aligned}\beta^i \leq \beta^j &\Leftrightarrow \frac{\alpha^i\sigma}{\alpha^i\sigma + (1-\alpha^i)\rho} \leq \frac{\alpha^j\sigma}{\alpha^j\sigma + (1-\alpha^j)\rho} \\ &\Leftrightarrow \alpha^i\sigma\rho \leq \alpha^j\sigma\rho \Leftrightarrow \alpha^i \leq \alpha^j,\end{aligned} \tag{25}$$

where the nonnegativeness of ρ_k and σ_k has been used in the second and third equivalence. ∎

We are now ready to state the main result for the nonsingular case. The following lemma shows that λ_k^* can be considered instead of y_k^* when checking for local optimality and for deciding which index that should be removed from $\mathcal{W}$.

Lemma 7: If $M_k M_k^T$ is nonsingular

1) $y_k^* \geq 0 \Leftrightarrow \lambda_k^* \geq 0$.
2) $\underset{h \in \mathcal{W}_k : [y_k^*]_h < 0}{\operatorname{argmin}} \alpha_k^h = \underset{h \in \mathcal{W}_k : [\lambda_k^*]_h < 0}{\operatorname{argmin}} \beta_k^h$.

Proof: First, since $M_k M_k^T$ is nonsingular we have that $\sigma_k > 0$ which gives

$$\mathcal{H}_k \triangleq \{h \in \mathcal{W}_k : [y_k^*]_h < 0\} = \{h \in \mathcal{W}_k : [\lambda_k^*]_h < 0\},$$

since $\frac{y_k^*}{\sigma_k} = \lambda_k^*$. I.e., the same indices of y_k^* and λ_k^* will be negative, and these components are given by the set $\mathcal{H}_k$, which proves 1). Next, $[y_k^*]_h < 0$ inserted into (19) gives $\alpha_k^h \in [0,1), \forall h \in \mathcal{H}_k$. Hence, Lemma 6 gives the same ordering of $\{\alpha_k^h\}_{h\in\mathcal{H}_k}$ and $\{\beta_k^h\}_{h\in\mathcal{H}_k}$ which means that the same index will give a minimum. ∎

D. Singular case

$M_k M_k^T$ only becomes singular after a component is added to $\mathcal{W}$ in Algorithm 1. In this case, the solution to (5), $[y_k^*]_{\mathcal{W}_k}$, will be a singular eigenvector to $M_k M_k^T$ as is shown by the following lemma.

Lemma 8: If $M_k M_k^T$ becomes singular in Algorithm 1, $M_k M_k^T [y_k^*]_{\mathcal{W}_k} = 0$ and $\sigma_k = 0$.

Proof: If $M_k M_k^T$ is singular, $\exists \tilde{\lambda}_k \neq 0, \tilde{\lambda}_k \in \mathbb{R}^m$ such that $M_k M_k^T [\tilde{\lambda}_k]_{\mathcal{W}_k} = 0$, $[\tilde{\lambda}_k]_{\bar{\mathcal{W}}_k} = 0$. Now, define $\delta_k \triangleq d^T \tilde{\lambda}_k$. Then $y_k^* = -\frac{\gamma}{\delta_k}\tilde{\lambda}_k$ leads to the objective function of (11) being zero and hence, since norms are nonnegative, this is a minimizer of (11). Inserting this y_k^* into the definition of σ_k gives $\sigma_k = 0$ by construction.

What remains to prove is that $\delta_k \neq 0$, so y_k^* from above is well-defined. Since $M_k M_k^T$ only becomes singular after an addition to $\mathcal{W}$, let i be the component that was added to $\mathcal{W}$ at iteration $k-1$, i.e., $\mathcal{W}_k = \mathcal{W}_{k-1} \cup \{i\}$. From (15), μ_{k-1} is given as

$$M M_{k-1}^T \lambda_{k-1}^* + d = \mu_{k-1}. \tag{26}$$

Multiplying this equation with $\tilde{\lambda}_k^T$ from the left gives

$$\tilde{\lambda}_k^T M M_{k-1}^T \lambda_{k-1}^* + \tilde{\lambda}_k^T d = [\tilde{\lambda}_k]_i [\mu_{k-1}]_i \Leftrightarrow \tag{27a}$$

$$\tilde{\lambda}_k^T d = [\tilde{\lambda}_k]_i [\mu_{k-1}]_i, \tag{27b}$$

where we have recalled the partitions of λ^* and μ from (16a) and (16b), respectively. Furthermore, we have also used that $\tilde{\lambda}_k$ is a singular eigenvector of M^T by construction.

Since i was added to $\mathcal{W}$ at iteration $k-1$, $[\mu_{k-1}]_i < 0$. Furthermore, $[\tilde{\lambda}_k]_i \neq 0$ since $[\tilde{\lambda}_k]_{\mathcal{W}_k}^T M_k = 0$ and $[\tilde{\lambda}_k]_i = 0$ would imply that $[\tilde{\lambda}_k]_{\mathcal{W}_{k-1}}^T M_{k-1} = 0$, which is impossible since $M_{k-1} M_{k-1}^T$ was nonsingular, hence, $\delta_k \triangleq \tilde{\lambda}_k^T d \neq 0$. ∎

Remark 3: We will, without loss of generality, assume that $\tilde{\lambda}_k$ is such that $\delta < 0$. This is valid since $M_k M_k^T \tilde{\lambda}_k = 0 \implies M_k M_k^T (-\tilde{\lambda}_k) = 0$. Hence, we can always change the sign of δ_k by changing the sign of $\tilde{\lambda}_k$.

Using Lemma 8 together with (20) gives the following update of λ in the singular case

$$\begin{aligned}\lambda_{k+1} =& \frac{y_k + \alpha_k(y_k^* - y_k)}{\gamma + d^T(y_k + \alpha_k(y_k^* - y_k))} \\ =& \frac{(1-\alpha_k)y_k + \alpha_k y_k^*}{(1-\alpha_k)\rho_k} = \lambda_k + \frac{\alpha_k}{(1-\alpha_k)\rho_k} y_k^* \\ =& \lambda_k - \frac{\alpha_k \gamma}{(1-\alpha_k)\rho_k \delta_k}\tilde{\lambda}_k = \lambda_k + \tilde{\beta}_k \tilde{\lambda}_k,\end{aligned} \tag{28}$$

where $\sigma_k = 0$ is used in the second equality, $y_k^* = -\frac{\gamma}{\delta_k}\tilde{\lambda}_k$ is used in the fourth equality and $\tilde{\beta}_k \triangleq -\frac{\alpha_k \gamma}{(1-\alpha_k)\rho_k \delta_k}$ has been defined in the last equality. Similar to α_k^i and β_k^i, we introduce the definition

$$\tilde{\beta}_k^i \triangleq -\frac{\gamma}{\rho_k \delta_k} \cdot \frac{\alpha_k^i}{(1-\alpha_k^i)} = -\frac{[\lambda_k]_i}{[\tilde{\lambda}_k]_i} \tag{29}$$

to denote the step length which results in $[\lambda_{k+1}]_i = 0$ when a step is taken in the direction $\tilde{\lambda}_k$ during iteration k.

Remark 4: The definition of $\tilde{\beta}_k^i$ in (29) is well-defined since $\rho_k > 0, \delta_k < 0$ and $\alpha_k^i \in [0,1)$. $\delta_k < 0$ has been established in Lemma 8, $\rho_k > 0$ follows from $M_k M_k^T$ only becoming singular after a constraint has been added to $\mathcal{W}_k$, which implies that $y_k = y_{k-1}^* \implies \rho_k = \sigma_{k-1} > 0$ since $M_{k-1} M_{k-1}^T$ was nonsingular.

Analogous to Lemma 6 for the nonsingular case, we establish the following properties for $\tilde{\beta}_k^i$

Lemma 9: If $\alpha_k^i, \alpha_k^j \in [0,1)$ then

$$\alpha_k^j \leq \alpha_k^i \Leftrightarrow \tilde{\beta}_k^j \leq \tilde{\beta}_k^i \tag{30}$$

Proof: Using the definition of $\tilde{\beta}_k^i$ in (29)

$$\begin{aligned}\tilde{\beta}_k^j \leq \tilde{\beta}_k^i &\Leftrightarrow -\frac{\gamma}{\rho_k\delta_k} \cdot \frac{\alpha_k^j}{(1-\alpha_k^j)} \leq -\frac{\gamma}{\rho_k\delta_k} \cdot \frac{\alpha_k^i}{(1-\alpha_k^i)} \\ &\Leftrightarrow (1-\alpha_k^i)\alpha_k^j \leq (1-\alpha_k^j)\alpha_k^i \Leftrightarrow \alpha_k^j \leq \alpha_k^i,\end{aligned} \tag{31}$$

where $-\frac{\gamma}{\rho_k\delta_k} > 0$ and $\alpha_k^i, \alpha_k^j \in [0,1)$ have been used in the second equivalence. $-\frac{\gamma}{\rho_k\delta_k} > 0$ follows from Remark 4. ∎

The following lemma is analogous to Lemma 7 but for the singular case. It shows that $\tilde{\lambda}_k$ can be considered instead of y_k^* when removing indices from $\mathcal{W}$ in the singular case.

Lemma 10: If $M_k M_k^T$ is singular

1) $y_k^* \geq 0 \Leftrightarrow \tilde{\lambda}_k \geq 0$.

2) $\underset{h\in\mathcal{W}_k:[y_k^*]_h<0}{\operatorname{argmin}} \alpha_k^h = \underset{h\in\mathcal{W}_k:[\tilde{\lambda}_k]_h<0}{\operatorname{argmin}} \tilde{\beta}_k^h$.

Proof: If $M_k M_k^T$ is singular we have from Lemma 8 that $y_k^* = -\frac{\gamma}{\delta_k}\tilde{\lambda}_k$, hence,

$$\mathcal{H}_k \triangleq \{h \in \mathcal{W}_k : [y_k^*]_h < 0\} = \{h \in \mathcal{W}_k : [\tilde{\lambda}_k]_h < 0\},$$

since $-\frac{\gamma}{\delta_k} > 0$. I.e, the same indices of y_k^* and $\tilde{\lambda}_k$ will be negative, and these components are given by the set $\mathcal{H}_k$, which proves 1). Next, $[y_k^*]_h < 0$ inserted into (19) gives $\alpha_k^h \in [0, 1), \forall h \in \mathcal{H}_k$, hence, Lemma 9 can be used and gives the same ordering of $\{\alpha_k^h\}_{h\in\mathcal{H}_k}$ and $\{\tilde{\beta}_k^h\}_{h\in\mathcal{H}_k}$ which means that the same index will give a minimum. ∎

Remark 5: From Farkas' lemma it is necessary for at least one component of y_k^*, and hence of $\tilde{\lambda}_k$, to be negative if the QP is feasible, cf. Theorem 1 in [11]. Therefore, since we assume that (1) is feasible, at least one constraint will be removed from $\mathcal{W}_k$ at iteration k if $M_k M_k^T$ is singular, regaining nonsingularity of $M_{k+1} M_{k+1}^T$.

IV. Certification of NNLS algorithm

We will now use the properties of Algorithm 1, which have been established in Section III, to certify its iteration complexity for all parameters in Θ_0. This is done by proving that Algorithm 1 produces the same working-set sequence as a standard primal active-set method, e.g., [7][20][21], applied to the dual of (1).

After this equivalence has been established, the result from [15] can be used to determine the working-set changes and the number of iterations any $\theta \in \Theta_0$ produces, which is done by applying the certification method presented in [15] to the dual of (1). The dual problem of (1), using the definitions of M and $d(\theta)$ from (2), is given by

$$\underset{\lambda\geq 0}{\text{minimize}} \quad \frac{1}{2}\lambda^T M M^T \lambda + d^T(\theta)\lambda, \tag{32}$$

Algorithm 2 presents a standard primal active-set method which is applied to solve (32). A complexity certification method for Algorithm 2, which determines the working-set sequences that are produced by Algorithm 2 for every $\theta \in \Theta_0$, is presented in [15]. We are now ready to state the main result of this paper, namely that Algorithm 1 will produce the same working-set sequence as Algorithm 2, for which there exists a certification method to determine *exactly* which working-set sequence any parameter will generate [15].

Algorithm 2 A standard primal active-set quadratic programming method applied to (32) [15].

$v \leftarrow \mathcal{L}^{-T}(f_\theta\theta + f)$; $\quad d \leftarrow (b + W\theta) + Mv$
$k \leftarrow 1, \mathcal{W}_k \leftarrow \emptyset; \lambda_k \leftarrow 0$;
while true **do**
  if $M_k M_k^T$ is singular **then go to** step 16
  $\lambda_k^* \leftarrow$ solution to KKT-system (15)
  if $\lambda_k^* \geq 0$ **then**
    $\mu_k \leftarrow M(M_k^T[\lambda_k^*]_{\mathcal{W}_k}) + d$
    if $\mu_k \geq -\epsilon$ **then**
      return $\lambda^*, x^* \leftarrow -\mathcal{L}^{-1}(M_k^T\lambda_k^* + v), \mathcal{W}_k$
    else $i \leftarrow \underset{i\in\bar{\mathcal{W}}_k}{\arg\min}\, [\mu_k]_i$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \cup \{i\}$
      $\lambda_{k+1} \leftarrow \lambda_k^*$
  else $l \leftarrow \underset{h\in\mathcal{W}:[\lambda_k^*]_h<0}{\arg\min} \{\frac{[\lambda_k]_h}{[\lambda_k - \lambda_k^*]_h}\}$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \setminus \{l\}$
    $\beta_k \leftarrow \frac{[\lambda]_l}{[\lambda - \lambda_k^*]_l}$; $\quad \lambda_{k+1} \leftarrow \lambda_k + \beta_k(\lambda_k^* - \lambda_k)$
  $k \leftarrow k + 1$
end while
$\tilde{\lambda}_k \leftarrow$ solution to $M_k M_k^T \tilde{\lambda}_k = 0,\ d_k^T\tilde{\lambda}_k < 0$
if $\tilde{\lambda}_k \geq 0$ **then**
  return primal infeasible
else $l \leftarrow \underset{h\in\mathcal{W}:[\tilde{\lambda}_k]_h<0}{\arg\min} \{-\frac{[\lambda_k]_h}{[\tilde{\lambda}_k]_h}\}$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \setminus \{l\}$
  $\tilde{\beta}_k \leftarrow -\frac{[\lambda_k]_l}{[\tilde{\lambda}_k]_l}$; $\quad \lambda_{k+1} \leftarrow \lambda_k + \tilde{\beta}_k\tilde{\lambda}_k$
  $k \leftarrow k + 1$
  go to step 5

Theorem 1: Let $\tilde{k}(\theta)$ be the number of iterations needed by Algorithm 2 to terminate and let $\{\tilde{\mathcal{W}}_k(\theta)\}_{k=1}^{\tilde{k}(\theta)}$ be the corresponding working-set sequence produced by Algorithm 2. Then Algorithm 1 terminates in $\tilde{k}(\theta)$ iterations and produces the working-set sequence $\{\tilde{\mathcal{W}}_k(\theta)\}_{k=1}^{\tilde{k}(\theta)}$, $\forall\theta \in \Theta_0$.

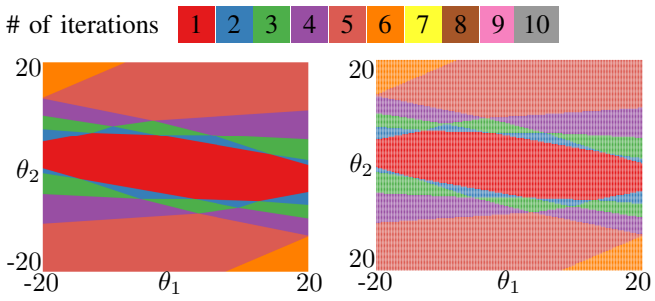
Proof: The theorem will be proven by using the properties derived in Section III to map an iteration of Algorithm 1 to an iteration of Algorithm 2. Table I summarizes the correspondence between each line of Algorithm 1 to lines of Algorithm 2 and which equation, Lemma or Remark proves each correspondence. Both the case when $M_k M_k^T$ is singular and nonsingular is shown in Table I. ∎

Alg. 1	Alg. 2	nonsingular	Alg. 2	singular
#1-3	#1-3	-	#1-3	-
#4	#5	(15)	#16	Lemma 8
#5	#6	Lemma 7	#17	Lemma 10
#6-10	#7-11	Lemma 3	#18	Remark 5
#11-13	#12-13	Lemma 7	#19-20	Lemma 10
#14	#14	-	#21	-
#15	#9	(6)	Impossible if primal feasible	

TABLE I: Mapping from Algorithm 1 to Algorithm 2

A direct consequence of Theorem 1 is that the worst-case number of iterations when (1) is solved with Algorithm 1 can be certified by applying the complexity certification method from [15] to the dual mpQP given in (32).

Remark 6: There are numerous active-set algorithms that are equivalent, cf. [22]. As is discussed in [15], Algorithm 2 is equivalent to, e.g., Dantzig's active-set method for QPs [21]



(a) Certification (b) Simulation

Fig. 1: Number of iterations determined by: (a) Applying the certification method presented in [15] to the dual mpQP; (b) Executing Algorithm 1 over a two-dimensional grid in the parameter space. $\theta_i = 0$ for $i \neq 1, 2$.

applied to the dual QP. Theorem 1 together with this equivalence explain the empirical observation in [11] that Algorithm 1 produces the same number of iterations as Dantzig's method.

V. Numerical Example

To exemplify the complexity certification method for Algorithm 1, an mpQP originating from the application of MPC to an inverted pendulum was considered. The resulting mpQP had the dimensions, $m = 10$, $p = 8$, and $n = 5$. Further details about the problem are given in [14].

The certification method was compared with results obtained by drawing samples from Θ_0 and executing Algorithm 1 on the resulting QPs. Figure 1 compares such simulations of Algorithm 1 for θ taken on a grid on a 2-dimensional subspace of Θ_0, with a slice of the partition, corresponding to the same subspace, obtained by applying the certification method from [15] to the dual of the mpQP. As Theorem 1 predicts, the resulting number of iterations is equal for the simulation and the certification. In addition, 10^8 random samples of the entire Θ_0 were taken and Algorithm 1 was applied to the resulting mpQPs. As before, these simulation results were compared with the results from applying the certification method from [15] to the dual of the mpQP. Again, as predicted by Theorem 1, both the simulation and the certification resulted in the same number of iterations.

The computation time required for the complexity certification of the inverted pendulum example was 7.6 seconds when executed on an Intel® 2.7 GHz i7-7500U CPU. For more details about the certification method itself, such as complexity, see [15].

VI. Conclusion

This paper has proposed a complexity certification method for a simple and efficient QP method. The complexity certification was done by relating the QP method to a standard primal active-set method applied to the dual of the QP, allowing the complexity certification method in [15] to be directly applicable. Future research includes certifying the complexity of the extended method presented in [17], which improves the numerical stability of the QP method considered in this paper.

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