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Joint Estimation of States and Parameters in Stochastic SIR Model

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Abstract— **The classical SIR model is a fundamental building block in most epidemiological models. Despite its widespread use, its properties in filtering and estimation applications are much less well explored. Independently of how the basic SIR model is integrated into more complex models, the fundamental question is whether the states and parameters can be estimated from a fusion of available numeric measurements. The problem studied in this paper focuses on the parameter and state estimation of a stochastic SIR model from assumed direct measurements of the number of infected people in the population, and the generalisation to other measurements is left for future research. In terms of parameter estimation, two components are discussed separately. The first component is model parameter estimation assuming that the all states are measured directly. The second component is state estimation assuming known parameters. These two components are combined into an iterative state and parameter estimator. This iterative method is compared to a straightforward approach based on state augmentation of the unknown parameters. Feasibility of the problem is studied from an information-theoretic point of view using the Cramér Rao Lower Bound (CRLB). Using simulated data resembling the first wave of Covid-19 in Sweden, the iterative method outperforms the state augmentation approach.**

Index Terms— **SIR Epidemic Model, Iterative Parameter and State Estimation, Cramér Rao Lower Bound, Bayesian Filtering and Smoothing.**

I. INTRODUCTION

Mathematical modelling gives a way to investigate epidemics more quantitatively and makes it easier for decision makers to design social policies, such as social distance, vaccination, quarantine, lockdown, etc [1]. The compartmental model is one of the widely used model structures in epidemiology. The Susceptible-Infectious-Recovered (SIR) model [2] is a type of widely used compartmental epidemics model, which is composed of susceptible, infectious, and recovered compartments, respectively. A brief explanation of the SIR model with a notation and methodological fusion framework similar to this paper can be found in [3]. In the SIR model, the dynamic trajectory is significantly impacted by two of the parameters, i.e., the spreading and recovering rates [4]. The basic reproductive number, R_0, is defined as the ratio between these two parameters, and this number is a well-known index for epidemics prediction and intervention. However, it is usually hard to measure the spreading and recovering rates in most real-world cases. Therefore, it is important to estimate these parameters from the data.

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Parameter identifiability is thus an important property of the model, where several approaches have been suggested in the literature. The first category mainly deals with deterministic formulations and uses the Least Squares (LS) method for identification. Some methods are proposed in [5], [6], [7] for the SIR model, and some others for extended models, Susceptible-Infectious-Recovered-Deceased (SIRD) model [8], [9], and Susceptible-Exposed-Infectious-Recovered (SEIR) model [10]. The performance is evaluated from the curve fitting aspect, and the parameter estimation is assumed to be solved perfectly if it has a small curve fitting error. However, the curve fitting criterion is not enough to analyse the model identification problem [3]. Some other works investigate the stochastic SIR model. [11], [12] discuss the method to construct a stochastic model. [13] solves the problem using Markov Chain Monte Carlo (MCMC) methods. Some works involve multiple measurement data. [14], [15] incorporate state augmentation method to estimate the model parameters. [16], [17] treat state and parameter separately and solve them iteratively. In [16], they use primal-dual interior point method for optimisation, whereas we use weighted least squares (WLS), which is simpler and has a lower complexity. The parameter in their method is the covariance of noise, which is different from us. In [17], complicated gradient and Hessian matrix are compulsory, whereas we do not need them. The common difference compared with [16], [17] is evaluation criterion. We evaluate the parameter estimation accuracy, whereas they use cruve fitting. Besides, we add the parameter identifiability issue which is not contained in their papers.

The main contribution of this paper is that we propose an iterative method to estimate both the parameter and states in stochastic SIR models. We assume access to the number of infected people per day. The assumed data may come from prevalence tests which take some random selection of the population to check whether they are infected or not. Our method can be easily applied to the extended models, such as SIRD and SEIR, and other observed numerical data related to the infection, such as patients in hospital, in ICU, deaths or estimated immunity in the population.

In terms of parameter estimation, the current estimated state trajectory is used to formulate a WLS problem, and in terms of state estimation, an extended Kalman smoother (EKS) is implemented given the current estimation of parameter values. Compared to previous research which are based on iterative methods, the parameter optimisation is straightforward with WLS. The information content of the parameters is analysed using the CRLB, and a positive definite decreasing bound in time is a necessary condition for parameter observability. We illustrate the results using simulated data from the SIR model are tuned to give a good curve fit to the first wave of covid-19 in Sweden. We compare the proposed iterative algorithm to a straightforward EKS approach, where the state is augmented with the parameters.

The structure of this paper is the following: the stochastic SIR model is formulated in Section 2 with details about the model itself. In Section 3 and 4, the parameter and state estimation problems, respectively, are considered individually assuming the other one is given. Section 5 contains the state augmentation method, observability, iterative method, and numerical example. Section 6 contains the conclusion.

II. Model and Problem Description

A. Stochastic SIR Model

The SIR model is composed of the three compartments Susceptible, Infectious and Recovered. A stochastic version of the SIR model is given as:

$$\begin{aligned} \frac{ds(t)}{dt} &= -\lambda s(t)i(t) + s(t)i(t)w_s(t) \\ \frac{di(t)}{dt} &= \lambda s(t)i(t) - \gamma i(t) - s(t)i(t)w_s(t) + i(t)w_i(t) \\ \frac{dr(t)}{dt} &= \gamma i(t) - i(t)w_i(t). \end{aligned} \tag{1}$$

Here, $s(t)$ denotes the susceptible fraction of the population, which relates to the people who can be infected, $i(t)$ denotes the infected fraction of the population, while $r(t)$ denotes the recovered fraction of the population. Here, the parameter λ is the spreading rate, which denotes how fast the pandemic transmits. Its value could be interpreted as the average amount of people each infectious person can infect every day. Further, γ denotes the recovery rate, and the reverse of it, $\frac{1}{\gamma}$, could be interpreted as the number of days on average a person is infectious (not ill).

The vector $w(t) = \begin{bmatrix} w_s(t) & w_i(t) \end{bmatrix}^T$ is assumed to be Gaussian process noise with covariance matrix Q, where we assume that the small disturbances in the number of infected and recovered each day are independent, so $Q = \begin{bmatrix} q_s & 0 \\ 0 & q_i \end{bmatrix}$. The reason for this multiplicative form for how the noise enters the states is that the model uncertainty should be large when we have large values of $s(t)$ and $i(t)$, and the uncertainty should be small when these states are small.

In the current model (1), all the three variables should be non-negative and the sum of them represents the fraction of the whole population and must hence always be 1, which means:

$$\begin{aligned} s(t)+i(t)+r(t) &= 1 \\ s(t), i(t), r(t) &\in [0,1]. \end{aligned} \tag{2}$$

These constraint are crucial for generating simulated data, but not so important for filtering. Because the measurement provides enought information for confidence interval (CI) of states to be within the constraints.

B. Discrete Time Model

The continuous model (1) can be discretised using the Euler-Maruyama method. Based on (2), the third variable, $r(t)$, could be eliminated from the model because it can be calculated from the other two variables. The discrete-time model that will be used in the sequel is given by:

$$\begin{aligned} s_{k+1} &= s_k - \tau\lambda s_k i_k + s_k i_k v_{s,k} \\ i_{k+1} &= i_k + \tau\lambda s_k i_k - \tau\gamma i_k - s_k i_k v_{s,k} + i_k v_{i,k}, \end{aligned} \tag{3}$$

where τ is the time step, which can control the accuracy of discretisation. The subscript, k, is used as a shorthand of the quantity at time $k\tau$, e.g., $s_k = s(k\tau)$. For simplicity, τ is set to 1 day in the sequel.

C. Measurement Model

Many potential data sources are both publicly available and have been used in epidemiological studies. Possible measurements include the following ones:

$$y_k^{\text{prev}} = N i_k + e_k^{\text{prev}} \tag{4}$$

$$y_k^{\text{anti-body}} = N(1 - i_k - s_k) + e_k^{\text{anti-body}} \tag{5}$$

$$y_k^{\text{death}} = N\alpha i_{k-d^{\text{death}}} + e_k^{\text{death}} \tag{6}$$

$$y_k^{\text{ICU}} = N\beta i_{k-d^{\text{ICU}}} + e_k^{\text{ICU}} \tag{7}$$

$$y_k^{\text{hosp}} = N\delta i_{k-d^{\text{hosp}}} + e_k^{\text{hosp}}. \tag{8}$$

Here y_k denotes the measurement, and the superscript denotes what the measurement is, here exemplified with prevalence, anti-body test, mortality, ICU and hospitalisation. Since measurements are usually in the unit of people and not fractions of the populations, the measurements are here scaled with the number of people in the population N, which is assumed to be known. α denotes the fraction of death, which is Infectious Fatality Ratio (IFR). β denotes the fraction of ICU, and δ denotes the fraction of hospitalisation. The measurement noise e_k is assumed to be Additive White Gaussian Noise (AWGN). An important discrete parameter is the delay d between the day the infection occurs and when its consequence is observed. In this paper, we assume that we have access to daily observations of the infectious people y_k^{prev}, which does not include a delay. The reason is to focus on the fundamental filtering and estimation problem.

D. Model Example for Illustration

We will illustrate the results on simulated data. To make the data semi-realistic, we tune them to give a good fit to the first wave of Covid-19 in Sweden during the spring and summer of 2020, for which daily fatality data is available. The number of deaths per day have a weakly regular pattern because of delayed reporting. The common solution is to apply a moving average filter on the measurement,

$$\bar{y}_k^{\text{death}} = \frac{1}{7}\sum_{j=k-3}^{k+3} y^{\text{death}}(j). \tag{9}$$

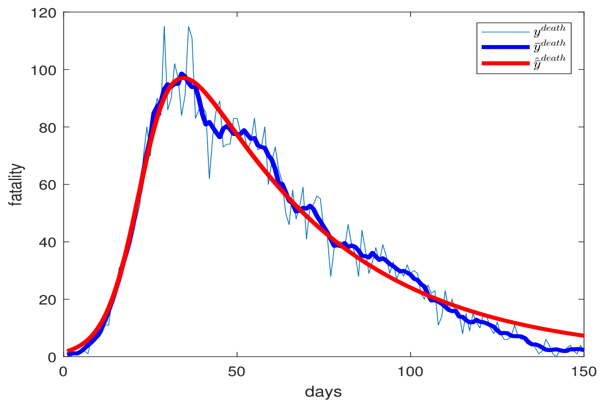

Fig. 1. Measured, averaged and estimated fatality data. The thin blue curve is the reported number of daily deaths, while the thick blue curve is $\bar{y}_k^{\text{death}}$ and the red curve is simulated fatality data without process noise using parameters in Table I.

TABLE I
PARAMETER VALUES IN SIMULATION AND FILTERING

$[\gamma,\lambda]^T$	$[s_0,i_0]^T$	$\left[s_0^f, i_0^f\right]^T$
$\begin{bmatrix} 0.0237 \\ 0.2256 \end{bmatrix}$	$\begin{bmatrix} 0.9891 \\ 0.0109 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
Q_{sim}	Q_{fs}	R
$\begin{bmatrix} 0.02 & 0 \\ 0 & 0.006 \end{bmatrix}$	$2Q_{\text{sim}}$	2.7524×10^5

For the ground truth of parameter values in the simulation in later sections, we choose the parameter values which minimise the LS cost between fitted and the averaged data $\bar{y}_k^{death}$. A comparison between measured and simulated fatality data is given in Fig. 1. We have used the value $\alpha = 1.4583 \times 10^{-5}$ for the IFR parameter. The reason for this value is that we plan to get a small curve fitting error to get a value for parameter λ, γ and initial value of state for simulation, and α is not further used in this paper. The parameter values used for simulation and filtering are summarised in Table I.

In Table I, s_0 is the initial value of the susceptible fraction, which can be calculated as $s_0 = 1 - i_0$. s_0^f and i_0^f are initial values in filtering in the following sections. R denotes the standard deviation of measurement noise for infection data in (4), and the reason for such a large number is because the population N, is 10^7. Q_{sim} and Q_{fs} are standard deviations for process noise used in the later section.

E. Problem Description

As stated before, the goal is to jointly estimate the state trajectory $x_{1:T}$ and parameters $\theta = (\gamma, \lambda)^T$ from the observations $y_{1:T}$ on an interval $k \in [0, T]$. First, we will break down the joint estimation into the simpler problems of estimating θ from an estimated state trajectory $x_{1:T}$ with known uncertainty, followed by a section on estimating $x_{1:T}$ from $y_{1:T}$ and known θ.

III. PARAMETER ESTIMATION

If the state sequence $x_{1:T}$ is known, a WLS problem can be formulated for the model described in (3) and (4) rewritten as the following linear regression

$$d_{k+1} = \phi_{k+1}\theta + w_k. \tag{10}$$

Here, the virtual measurement d_{k+1} and regression vector ϕ_{k+1} are defined as:

$$d_{k+1} = \begin{bmatrix} s_{k+1} - s_k \\ i_{k+1} - i_k \end{bmatrix} \qquad \phi_{k+1} = \begin{bmatrix} -s_k i_k & 0 \\ s_k i_k & -i_k \end{bmatrix}. \tag{11}$$

The noise term, w_k, in (10), has the covariance matrix:

$$\Sigma_k = \begin{bmatrix} q_s^2 s_k^2 i_k^2 & -q_s^2 s_k^2 i_k^2 \\ -q_s^2 s_k^2 i_k^2 & q_s^2 s_k^2 i_k^2 + q_i^2 i_k^2 \end{bmatrix}. \tag{12}$$

The parameter can now be estimated using WLS:

$$\hat{\theta} = \arg\min_{\theta} \sum_{k=0}^{T-1} \|d_{k+1} - \phi_{k+1}\theta\|^2_{\Sigma_k^{-1}}, \tag{13}$$

where $\|(\cdot)\|_P^2 = (\cdot)^T P(\cdot)$. The closed-form solution is given by

$$\hat{\theta} = \left(\sum_{k=0}^{T-1} \phi_{k+1}^T \Sigma_k^{-1} \phi_{k+1}\right)^{-1} \left(\sum_{k=0}^{T-1} \phi_{k+1}^T \Sigma_k^{-1} d_{k+1}\right), \tag{14}$$

and the corresponding estimation covariance matrix is

$$\text{Cov}(\hat{\theta}) = \left(\sum_{k=0}^{T-1} \phi_{k+1}^T \Sigma_k^{-1} \phi_{k+1}\right)^{-1}. \tag{15}$$

In practice, it is impossible to get perfect measurements of i_k and s_k. In our later sections, their estimated values will be used. In this case, an error for each state is inevi and causes an errors in variable (EIV) problem. Total least squares (TLS) could be used to solve this [18]. However, TLS assumes all error terms are i.i.d., which cannot be satisfied here. By computing the weight matrix, Σ_k^{-1}, we find that it is sensitive to estimation error when s_k and i_k are small. Regularisation is used to mitigate this. The regularised weight matrix is

$$\Sigma_{k,re}^{-1} = (\Sigma_k + \eta \mathbf{I}_{2\times 2})^{-1}, \tag{16}$$

where $\mathbf{I}$ is the identity matrix. With proper tuning of the regularisation parameter η, WLS will be more robust to errors in (14).

IV. STATE ESTIMATION

Given a parameter vector $\theta = (\lambda, \gamma)^T$, the model (3) and (4) is a standard stochastic nonlinear state space model where the extended Kalman filter (EKF) and EKS apply.

Observability for nonlinear models depends on the state trajectory. One way to monitor observability is to study the CRLB for a given state trajectory. Since this is a lower bound on the achievable estimation error covariance matrix, a necessary condition for parameter observability is that the CRLB is positive definite and decreases when more measurements become available. Another indication is provided by just monitoring the covariance matrix from the EKF or EKS.

We exemplify the covariance matrix using simulated data of the model with parameters as given in Table I. Figure 2 compares the root mean square error (RMSE) to the CI computed from the covariance matrix for each state for the EKF and EKS, respectively. The solid line shows the average

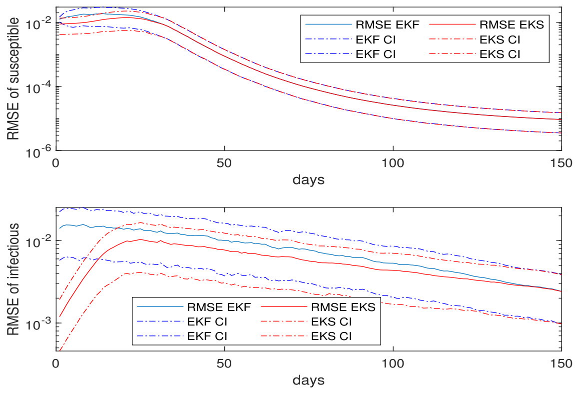

Fig. 2. RMSE of state estimation with EKF and EKS, model (3) and (4), and parameters in Table I are used. Q_{sim} is used for data generation. s_0^f, i_0^f and Q_{fs} are used for state estimation. R is used for data generation and state estimation. Solid line is the average RMSE over the MC runs, the dashed lines show the CI with respect to the noise realisations in the MC runs.

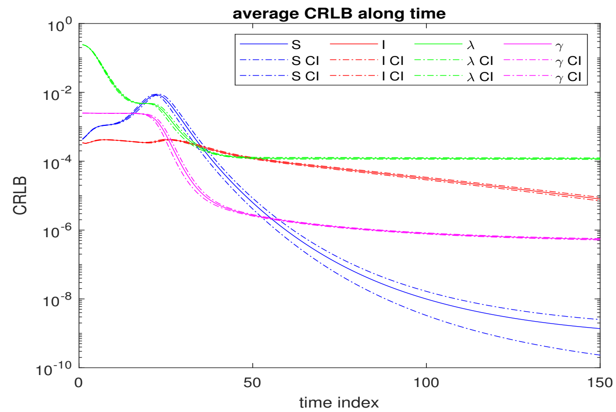

Fig. 3. CRLB for filtering with augmented SIR model. Solid lines are mean value of CRLB among simulations for each variable, dashed lines are 95% CI. The data is simulated with parameter values in Table I.

RMSE over the MC runs, while the dashed lines show the CI on RMSE from the MC simulations (thus showing the variability of the estimated estimation variance). The CI decreases over time, so the influence of the realisation of the process and measurement noise decreases over time. We can in particular note that the smoother gains information about the early phases of the wave from the later data, where one can make better predictions of the initial amount of infected people i_0 at time $k = 0$.

V. Joint State and Parameter Estimation

In the previous section, state and parameter estimation is discussed separately assuming the other one is given. However, none of them is given in practice. Therefore, we need to estimate both of them together.

A. State Augmentation

A simple method to estimate the parameter and state simultaneously is state augmentation, where the parameters are included in the state vector [19], [20]. The augmented state space model can then be written in the following form

$$\begin{aligned} x_{k+1,a} &= f_a(x_{k,a}) + g_a(x_{k,a})v_{k,a} \\ y_{k,a} &= h_a(x_{k,a}) + e_k, \end{aligned} \tag{17}$$

which is for convenience a bit more compact that the state space model before. Here a denotes augmentation, and

$$x_{k,a} = \begin{bmatrix} s_k & i_k & \lambda_k & \gamma_k \end{bmatrix}^T$$
$$v_{k,a} = \begin{bmatrix} v_{s,k} & v_{i,k} \end{bmatrix}^T$$

$$f_a(x_{k,a}) = \begin{bmatrix} s_k - \lambda_k s_k i_k \\ i_k + \lambda_k s_k i_k - \gamma_k i_k \\ \lambda_k \\ \gamma_k \end{bmatrix} \quad g_a(x_{k,a}) = \begin{bmatrix} s_k i_k & 0 \\ -s_k i_k & i_k \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$h_a(x_{k,a}) = N i_k.$$

the EKS can then be applied in an attempt to solve the joint parameter and state estimation problem.

B. Identifiability Issue for Parameters

Similar to approach to analyse the observability for the state estimation discussed in Section IV, we use the CRLB to investigate identifiability for parameters. The parameter is identifiable if, with model (17), we have:

$$P^C_{T|T} \leq P_{0|0}, \tag{18}$$

where $P_{0|0}$ is the covariance matrix of the initial state without any measurement, and $P^C_{T|T}$ is the CRLB of the estimated state at the last time step. The initial value for CRLB is set as $P^C_{0|0} = P_{0|0}$. The reason for using (18) is that the estimation of λ and γ at the last time step can be used. Therefore, (18) will be enough.

The parametric CRLB is used because of its simplicity, and it is computed as [21]:

$$\begin{aligned} P^C_{k+1|k} &= J^C_{x,k} P^C_{k|k} J^{C,T}_{x,k} + J^C_{w,k} Q_{a,k} J^{C,T}_{w,k} \\ P^C_{k+1|k+1} &= P^C_{k+1|k} - P^C_{k+1|k} H^{C,T}_k (H^C_k P^C_{k+1|k} H^{C,T}_k + R)^{-1} H^C_k P^C_{k+1|k}, \end{aligned} \tag{19}$$

where $J^C_{x,k}$ is the Jacobian matrix of the state transition model with respect to $x_{k,a}$ computed at the ground truth. $J^C_{w,k}$ is the Jacobian matrix of the state transition model with respect to process noise computed at the ground truth. H^C_k is the Jacobian matrix of measurement model with respect to $x_{k,a}$ at ground truth. The CRLB curve is given in Fig. 3.

As it is shown in Fig. 3, given the measurement data, the CRLB for each state and parameter is decreasing with time after the peak. This is a valid motivation for the identifiability of parameters.

C. Iterative Parameter and State Estimation

We propose to estimate state and parameter simultaneously using an iterative method. It starts from initial parameter values λ^0 and γ^0, and uses them to estimate the state with EKS. The estimated states can be represented as $s^1_{0:T}$ and $i^1_{0:T}$, where T denotes data length. With $s^1_{0:T}$ and $i^1_{0:T}$, the notations in (11) and (12) can be constructed. (14) can be used to solve the parameter estimation part afterward to get a new parameter value, λ^1 and γ^1. Then we can go back to state estimation with λ^1 and γ^1 for the next iteration and repeat until we meet the maximum iteration number condition. The implementation details are in Algorithm 1.

Algorithm 1 Iterative Parameter and State Estimation

Input: Initial parameter values as λ^0, γ^0, mean and covariance of the initial states s_0 and i_0, the maximum iteration number K.

for $j = 1:K$ **do**

Do EKS with model (3) and (4) using parameters λ^{j-1} and γ^{j-1} to get $s^j_{0:T}$ and $i^j_{0:T}$.

Construct d_j, ϕ_j and Σ_j using (11) and (12)

Use (14) to get γ^j and λ^j with d_j, ϕ_j and Σ_j

end for

Do EKS to estimate s^s_0 and i^s_0 with the parameters after the last iteration.

Output: λ^K and γ^K as parameter estimation, s^s_0 and i^s_0 and their covariance matrix as initial state estimation.

In Algorithm 1, λ^{j-1} and γ^{j-1} are parameter estimations after the $j-1$th iteration. $s^j_{0:T}$ and $i^j_{0:T}$ are estimated states in the jth iteration. s^f_0 and i^f_0 in Table I will be used to initialize the smoother in every iteration. In terms of the initial state value estimation, s^s_0 and i^s_0, should be estimated using the smoothing result after the last iteration. For each iterations, the initial state value of state estimation is fixed as s^f_0 and i^f_0. The reason is that it is found the state and parameter estimation is sensitive to the initial state value in EKS, and because the parameter value is far away from the ground truth in the first several iterations, the smoothing result for $[s^s_0, i^s_0]$ cannot be correct. If we replace the predefined value with the smoothed value, $[s^s_0, i^s_0]$, our algorithm will be divergent. For the stopping rule of the iterative method, we use maximum iteration criterion with an iteration number which can guarantee all simulation could be convergent.

D. Numerical Example

In this section, Monte Carlo simulations of the model (3) and (4) based on the values in Table I are used to evaluate the performance of the methods.

In the simulation, 500 simulations are done. The initial value of γ has the uniform distribution over the range $[0, 0.1]$, and the initial value of λ has the uniform distribution in $[0.1, 0.9]$. The process noise has standard deviation Q_{sim} for data generation, and Q_{fs} for state estimation as they are given in Table I. In terms of the initial covariance matrix for state estimation, $0.02^2 \times I_{2\times 2}$ is used. The reason for such a small value is that the initial value in the filter is $[1, 0]^T$, which is close to the ground truth. A comparison of the results using the iterative parameter and state estimation and the state augmentation of λ and γ methods, respectively, can be found in Fig. 4.

For the state augmentation method, three among the all 500 simulations resulted in that the estimated parameter

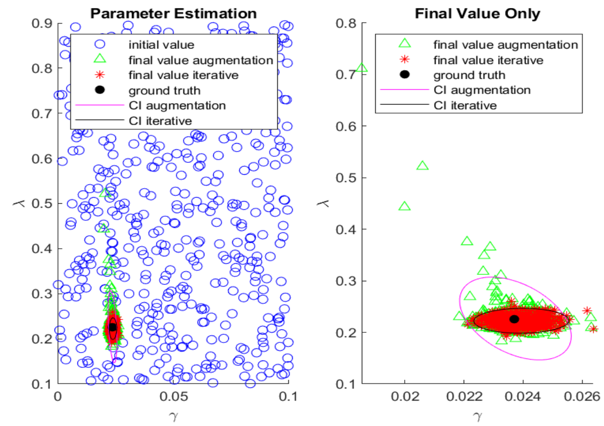

Fig. 4. Parameter estimation results for state augmentation and iterative method, respectively. 95% CI are computed through the covariance matrix of the parameters. The left figure contains initial and estimated parameter value. The right figure contains estimated value only for a clear look.

TABLE II
RMSE OF PARAMETERS IN TERMS OF DIFFERENT VALUE FOR REGULARISATION FACTOR.

η	LS	10^{-5}	10^{-6}	10^{-7}	10^{-8}
RMSE λ	0.0101	0.0096	0.0098	0.0101	0.0103
RMSE γ	9×10^{-4}	7×10^{-4}	7×10^{-4}	7×10^{-4}	7×10^{-4}

values are outside of the displayed area, whereas this did not happen for the iterative method. Both the actual estimation errors and the confidence ellipsoid are smaller for the iterative method.

In terms of regularisation factor in WLS as discussed in (16), Table II shows the RMSE of parameter esitmation with respect to the regularisation factor η. The purpose of regularisation factor is to avoid divergence. Our simulation shows that a too small value for η does not solve the problem of divergence. Table II shows that this parameter can be in a quite broad range without divergence in parameter estimation.

For the initial value of the state, the criterion about whether the initial value is estimated correctly can be rephrased as whether the CI of the estimated state covers the ground truth. We can use the chi-square test to detect whether the ground truth is inside the CI region. The test statistic is given by:

$$TS = \left(\begin{bmatrix} s^s_0 \\ i^s_0 \end{bmatrix} - \begin{bmatrix} s_0 \\ i_0 \end{bmatrix}\right)^T P_0^{-1,s} \left(\begin{bmatrix} s^s_0 \\ i^s_0 \end{bmatrix} - \begin{bmatrix} s_0 \\ i_0 \end{bmatrix}\right) \sim \chi^2(2). \quad (20)$$

Here, s^s_0 and i^s_0 are estimated initial states. s_0 and i_0 are the ground truth of the initial state. $P_0^{-1,s}$ is the covariance matrix for the initial state using EKS. The ground truth is covered in the CI region if the test statistic, is smaller than the threshold. The threshold is determined using the chi-square distribution. The result with different CI levels is given in Table III. For instance, the first number shows that 51.2% of the simulations for the iterative method ended up inside a confidence ellipsoid that corresponds to 50%. All in

TABLE III

COMPARISON OF HOW THE PARAMETER ESTIMATION ERRORS COMPARE TO THEIR THEORETICAL COVARIANCE.

CI level	50%	68%	90%	95%	99%
Iterative method	51.2%	67.6%	85.4%	88.8%	95.8%
State augmentation	16.7%	22.9%	34.1%	39.8%	48.6%

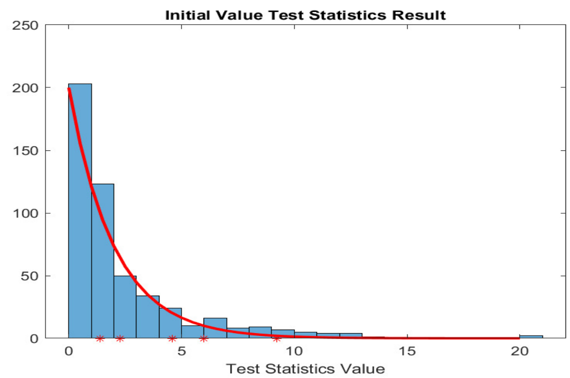

Fig. 5. Test statistics of initial value estimation with iterative method. The blue rectangulars are histogram of test statistics computed from (20). The red curve is chi-square distribution with freedom 2, the red crosses are threshold of different CI given in III.

all, the iterative methods outperforms the state augmentation approach both in terms of final parameter and initial state estimation.

A histogram of the test statistics is given in Fig. 5, which shows that test statistic is approximately chi-square distributed.

To summarise the simulations with focus on the parameter estimates:

- The confidence ellipsoid is smaller for the iterative method as shown in Fig. 4.
- The estimation errors are smaller for the iterative method as shown in Fig. 4.

VI. CONCLUSION

In this paper, the parameter and state estimation problem of the stochastic SIR model was investigated. The SIR model can approximate a given wave of a pandemic quite well, which was illustrated by the first wave in Sweden for Covid-19. The spreading and recovery rates are crucial to monitor. We proposed a method that iterates between state estimation using given parameters with EKS and parameter estimation using a known state trajectory with WLS.

We compared this method to the state augmentation. It was shown in a simulation study that the iterative method outperformed the augmentation method in terms of (i) smaller estimation errors and (ii) no divergence rate.

The state augmentation method was used to analyse the information content of the data using the CRLB. It was shown that the CRLB decreases over time for a given wave of the pandemic, which gives a necessary condition of parameter observability. The longer one waits into the wave, the better parameter estimates can be expected.

Future work will include the state estimation error into the WLS criterion, and vice versa, include the parameter estimation covariance into the state estimator. Another possible extension is to investigate Expectation Maximization (EM) algorithm, which is quite similar to the iterative method.

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