PSEUDOINVERSES OF SIGNED LAPLACIAN MATRICES

Angela Fontan and Claudio Altafini

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ANGELA FONTAN† AND CLAUDIO ALTAFINI‡

Abstract. Even for nonnegative graphs, the pseudoinverse of a Laplacian matrix is not an "ordinary" (i.e., unsigned) Laplacian matrix, but rather a signed Laplacian. In this paper, we show that the property of eventual positivity provides a natural embedding class for both signed and unsigned Laplacians, class which is closed with respect to pseudoinversion as well as to stability. Such class can deal with both undirected and directed graphs. In particular, for digraphs, when dealing with pseudoinverse-related quantities such as effective resistance, two possible solutions naturally emerge, differing in the order in which the operations of pseudoinversion and of symmetrization are performed. Both lead to an effective resistance which is a Euclidean metric on the graph.

Key words. Eventually exponentially positive matrix, signed graphs, signed Laplacian matrix, Moore-Penrose pseudoinverse, effective resistance

MSC codes. 05C22, 05C50, 05C12

1. Introduction.

The Laplacian matrix is a fundamental object used ubiquitously in many fields, such as graph theory, linear algebra, complex networks, dynamical systems and PDEs. It captures basic information on a graph, such as its connectivity and spectrum [12, 1] but also properties of a dynamical system living on the graph [30, 4, 7, 32]. Associated to the Laplacian is also a Laplacian pseudoinverse, typically a Moore-Penrose pseudoinverse, which has also been used extensively to describe graph-related quantities. For instance it is used to build an effective resistance matrix for the graph, a distance measure that exploits the analogy between graphs and electrical networks [24, 38, 20, 35, 15], and to compute hitting/commuting times in Markov chains [8, 31, 6, 37, 25]. It is also used to estimate the $\mathcal{H}_2$ norm in networked dynamical systems [39, 40, 26].

In this paper, we are interested in studying the properties of the Laplacian pseudoinverse, starting from the observation that even in the most common case (when the graph is undirected and has all nonnegative edges weights), the Laplacian pseudoinverse is *not* a Laplacian matrix. In fact, if we consider a connected graph with nonnegative edge weights, it is well-known that the Laplacian L is an M-matrix (i.e., a matrix with nonpositive off-diagonal entries, such that $-L$ is marginally stable, see below for proper definitions). It is also easy to show that the Laplacian pseudoinverse does not belong to the same class of matrices. Consider for instance the following example

$$L = \begin{bmatrix} 0.8 & -0.7 & -0.1 \\ -0.7 & 0.9 & -0.2 \\ -0.1 & -0.2 & 0.3 \end{bmatrix}. \tag{1.1}$$

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†Division of Decision and Control Systems, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden (angfon@kth.se). This research was performed while this author was with the Division of Automatic Control, Department of Electrical Engineering, Linköping University, SE-58183 Linköping, Sweden.

‡Corresponding author. Division of Automatic Control, Department of Electrical Engineering, Linköping University, SE-58183 Linköping, Sweden (claudio.altafini@liu.se).

Its pseudoinverse is

$$L^{\dagger} = \begin{bmatrix} 0.773 & 0.048 & -0.821 \\ 0.048 & 0.628 & -0.676 \\ -0.821 & -0.676 & 1.498 \end{bmatrix} \tag{1.2}$$

which has an anomalous sign in the (1,2) entry. Even though $L^{\dagger}$ is not an M-matrix, it nevertheless has most of the properties of an M-matrix, most notably a Perron-Frobenius property from which it follows that, just like it is for $-L$, the eigenpair formed by the 0 eigenvalue and the positive "all 1" eigenvector $\mathbb{1}$ is the dominant pair for $-L^{\dagger}$. In the linear algebra literature, such matrices are called *Eventually Exponentially Positive* (EEP) [28, 29, 23, 4].

It is easily shown through examples that similar arguments are valid if we extend our analysis to Laplacians associated to *signed graphs*. A signed graph is a graph whose edges can have both positive or negative weights [42]. Motivation for using signed graphs instead of "ordinary" (i.e., nonnegative weight) graphs comes e.g. from multiagent systems in which cooperative and antagonistic interactions coexist [2], small-disturbance angle stability analysis of microgrids [34], Jacobian linearization of Kuramoto oscillators beyond the phase cohesive set [14]. See also [16, 21] for other contexts of relevance. Of the two possible *signed Laplacians* that can be associated to a signed graph, in this paper we consider the so-called "repelling signed Laplacian" ([33], see next Section for a precise definition), whose main property is that it always has 0 as eigenvalue but it may or may not be stable. In [3] it is shown that the EEP property can be used to characterize stability of these signed Laplacians.

What is shown in this paper, instead, is that the pseudoinverse of an EEP signed Laplacian is an EEP signed Laplacian. In other words, unlike the class of "ordinary" Laplacians, the class of EEP signed Laplacians is closed with respect to pseudoinversion. In addition, for Laplacians that are also weight balanced (i.e., for which $\mathbb{1}$ is both the left and right dominant eigenvector) the class of EEP signed Laplacians is closed also with respect to stability. When we restrict further the class of signed Laplacians from weight balanced L to normal L, then we have that this class is also closed w.r.t. symmetrization, that is, the operation of taking the symmetric part. In particular the ensuing signed Laplacians and Laplacian pseudoinverses are both characterized by the fact of having a symmetric part which is positive semidefinite of corank 1. Such property is particularly useful in contexts such as the computation of effective resistance, which, being a distance, has to be symmetric.

It is also shown in the paper that the operations of symmetrization and of pseudoinversion do not commute: depending on the order in which they are applied one gets a different result. Of the two possibilities, one (symmetrization followed by pseudoinversion) is shown to be equivalent to the notion used in [41]; the other (pseudoinversion followed by symmetrization) is instead new and presented here for the first time. A shortcoming of the definition of [41] is that the "directedness" nature of a digraph is already lost before the pseudoinverse is computed, meaning that intrinsically non-symmetric quantities (like for instance computing hitting times in a Markov chain) become impossible to attain, while they are feasible with our new definition. When instead the pseudoinverse is used for computing intrinsically symmetric quantities like a graph distance, then both definitions are viable.

The rest of the paper is organized as follows: in Section 2 we introduce notation and preliminary material, while in Section 3 we review results on signed Laplacians from [3, 19]. In Section 4 we present the main results for the Laplacian pseudoinverse of signed graphs. Their application to the calculation of effective resistance is discussed in Section 5, while an outlook on other potential applications is provided in Section 6. Most of the proofs are put in the Appendices at the end of the paper.

A preliminary version of this work appears in the conference proceedings of CDC 2021 [18]. Apart from the proofs of the various results, which were missing in [18], also the material of Sections 5 and 6 is largely novel.

2. Preliminaries.

2.1. Linear algebraic preliminaries. Given a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, the (i,j)-th entry of A is denoted A_{ij} or $[A]_{ij}$. $A \geq 0$ means element-wise nonnegative, i.e., $a_{ij} \geq 0$ for all $i, j = 1,\dots,n$, while $A > 0$ means element-wise positive, i.e., $a_{ij} > 0$ for all $i, j = 1,\dots,n$. The spectrum of A is denoted $\mathrm{sp}(A) = \{\lambda_1(A),\dots,\lambda_n(A)\}$, where $\lambda_i(A)$, $i = 1,\dots,n$, are the eigenvalues of A. In this paper we use the ordering $\mathrm{Re}[\lambda_1(A)] \leq \mathrm{Re}[\lambda_2(A)] \leq \cdots \leq \mathrm{Re}[\lambda_n(A)]$, where $\mathrm{Re}[\lambda_i(A)]$ indicates the real part of $\lambda_i(A)$. The spectral radius of A is the smallest real nonnegative number such that $\rho(A) \geq |\lambda_i(A)|$ for all $i = 1,\dots,n$ and $\lambda_i(A) \in \mathrm{sp}(A)$. A matrix A is called *Hurwitz stable* if $\mathrm{Re}[\lambda_n(A)] < 0$, and *marginally stable* if $\mathrm{Re}[\lambda_n(A)] = 0$ and any eigenvalue $\lambda(A) \in \mathrm{sp}(A)$ with $\mathrm{Re}[\lambda(A)] = 0$ is a simple root of the minimal polynomial of A. A matrix A is called positive semidefinite (psd) if $x^T Ax = x^T \frac{A+A^T}{2} x \geq 0\ \forall x \in \mathbb{R}^n$ and it is called positive definite (pd) if $x^T Ax = x^T \frac{A+A^T}{2} x > 0\ \forall x \in \mathbb{R}^n \setminus \{0\}$. A matrix A is called irreducible if there does not exist a permutation matrix P s.t. $P^T AP$ is block triangular, that is $P^T AP \neq \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ where A_{11} and A_{22} are nontrivial square matrices. A matrix B is called a Z-matrix if it can be written as $B = sI - A$, where $A \geq 0$ and $s > 0$, and it is called an M-matrix if, in addition, $s \geq \rho(A)$, which implies that all the eigenvalues of B have nonnegative real part. If $s > \rho(A)$ then B is a nonsingular M-matrix and $-B$ is Hurwitz stable. If $s = \rho(A)$ then B is a singular M-matrix, and if A is irreducible then $-B$ is marginally stable. If A is a singular matrix, the Moore-Penrose pseudoinverse of A, denoted $A^\dagger$, is the unique $n \times n$ matrix that satisfies $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(A^\dagger A)^T = A^\dagger A$, and $(AA^\dagger)^T = AA^\dagger$. A singular matrix A is said to have index 1 if the range of A, $\mathcal{R}(A)$, and the kernel of A, $\mathcal{N}(A)$, are complementary subspaces, i.e., $\mathcal{R}(A) \cap \mathcal{N}(A) = 0$. For index 1 singular matrices, other types of inverses, like the Drazin inverse and the group inverse [27], coincide. A singular M-matrix has always index 1, see [27].

A matrix $A \in \mathbb{R}^{n\times n}$ is said to have corank d if the dimension of the kernel space of A, $\mathcal{N}(A)$, is d. A matrix is *normal* if it commutes with its transpose: $AA^T = A^T A$. A matrix A is said a *range symmetric* matrix ([27], also called "equal projector") if $\mathcal{N}(A) = \mathcal{N}(A^T)$ (and hence $\mathcal{R}(A) = \mathcal{R}(A^T)$). Range symmetric matrices generalize normal matrices, and like normal matrices have many equivalent characterizations, see [27]. For instance a range symmetric matrix A is such that A commutes with its Moore-Penrose pseudoinverse $A^\dagger$. If A is a range symmetric matrix, then $\exists U$ orthogonal such that $A = U \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} U^T$ with B nonsingular of dimension $r = \mathrm{rank}(A)$. Singular range symmetric matrices have index 1, and for them the Moore-Penrose pseudoinverse, the Drazin inverse and the group inverse coincide.

2.2. Signed graphs. Let $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E}, A)$ be the (weighted) digraph with vertex set $\mathcal{V}$ ($\mathrm{card}(\mathcal{V}) = n$), $\mathcal{E} = \mathcal{V} \times \mathcal{V}$, and adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$: $a_{ij} \in \mathbb{R} \setminus \{0\}$ iff $(j, i) \in \mathcal{E}$, where (j, i) represents a directed edge from node j to node i. A *signed* digraph $\mathcal{G}(A)$ is a digraph where each edge is labeled by a sign (i.e., $\mathrm{sign}\,(a_{ij}) = \pm 1$). To distinguish with the signed digraph case, the digraph $\mathcal{G}(A)$ is also called nonnegative or *unsigned* if $A \geq 0$. A node i is said to be linked to j if there exists an edge sequence $(j, i_1), (i_1, i_2), \ldots, (i_{s-1}, i_s), (i_s, i)$ that is picked from $\mathcal{E}$. We call $\mathcal{G}(A)$ strongly connected if each pair of nodes in $\mathcal{V}$ is linked to each other. For digraphs $\mathcal{G}(A)$ which are strongly connected and without self-loops, the matrix A is irreducible with null-diagonal. A digraph $\mathcal{G}(A)$ contains a rooted spanning tree if there exists a node (called root) such that any other node of the digraph is linked to it. The weighted in-degree and out-degree of node i are denoted $\sigma_i^{\text{in}} = \sum_{j=1}^n a_{ij}$ and $\sigma_i^{\text{out}} = \sum_{j=1}^n a_{ji}$, respectively. A digraph $\mathcal{G}(A)$ is *weight balanced* if in-degree and out-degree coincide for each node, i.e., $\sigma_i^{\text{in}} = \sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji} = \sigma_i^{\text{out}}$ for all $i = 1, \ldots, n$. The *signed Laplacian* of a graph $\mathcal{G}(A)$ is the (in general non-symmetric) matrix $L = [L_{ij}] \in \mathbb{R}^{n \times n}$, defined as

$$[L]_{ij} = \begin{cases} -a_{ij}, & j \neq i \\ \sum_{j=1}^n a_{ij} = \sigma_i^{\text{in}}, & j = i \end{cases} \tag{2.1}$$

Eq. (2.1) can be written in compact form as $L = \Sigma - A$, where $\Sigma = \text{diag}\left(\sigma_1^{\text{in}}, \ldots, \sigma_n^{\text{in}}\right)$. This definition of signed Laplacian corresponds to the so-called "repelling signed Laplacian" in the terminology of [33], terminology which allows to distinguish it from another signed Laplacian (referred to in [33] as "opposing signed Laplacian"), obtained replacing σ_i^{in} with $\sigma_i^{\text{in,abs}} = \sum_{j=1}^n |a_{ij}|$, see [33, 2]. If the graph $\mathcal{G}(A)$ is unsigned (i.e., $A \geq 0$), this definition equals the standard Laplacian matrix. While with a slight abuse of notation we use the letter L to indicate both a Laplacian and a signed Laplacian, we refer to a Laplacian (of an unsigned graph) as an *unsigned Laplacian* in this paper. By construction, the signed Laplacian L is a singular matrix with $\text{span}(\mathbb{1}) \in \mathcal{N}(L)$, where $\mathbb{1} \in \mathbb{R}^n$ is the vector of 1s; L is *weight balanced* if $L^T \mathbb{1} = L\mathbb{1} = 0$, i.e., if $\text{span}(\mathbb{1}) \in \mathcal{N}(L^T)$.

2.3. Kron reduction for undirected networks. Consider an undirected and connected graph $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E}, A)$ with adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. Let $\alpha \subset \{1, \ldots, n\}$ (with $\text{card}(\alpha) \geq 2$) and $\beta = \{1, \ldots, n\} \backslash \alpha$ be a partition of the node set $\mathcal{V} = \{1, \ldots, n\}$. After an adequate permutation of its rows and columns, the Laplacian L of the graph $\mathcal{G}(A)$ can be rewritten as $L = \begin{bmatrix} L[\alpha] & L[\alpha, \beta] \\ L[\beta, \alpha] & L[\beta] \end{bmatrix}$, where we denote $L[\alpha, \beta]$ the submatrix of L determined by the index sets α and β, and $L[\alpha] := L[\alpha, \alpha]$ the principal submatrix of L determined by the index set α. If $L[\beta]$ is nonsingular, the Schur complement of $L[\beta]$ in L is given by $L/L[\beta] := L[\alpha] - L[\alpha, \beta]L[\beta]^{-1}L[\beta, \alpha]$.

In the context of electrical networks, where α and β are referred to as boundary (or terminal) and interior nodes, this procedure is denoted Kron reduction (see e.g. [13, 15, 35]) and it yields a matrix $L_r := L/L[\beta]$, denoted Kron-reduced matrix, which is still a Laplacian of an undirected graph $\mathcal{G}_r$ (see [13] for details and properties of L_r in the case of unsigned networks). If $\mathcal{G}(A)$ is signed and undirected, L_r is a signed symmetric Laplacian matrix and, when α is chosen as the set of nodes incident to edges with negative weight, it is shown in [10] that $L[\beta]$ is positive definite and that L is psd of corank 1 if and only if L_r is psd of corank 1.

2.4. Eventual exponential positivity.

Definition 2.1. *A matrix $A \in \mathbb{R}^{n \times n}$ has the Perron-Frobenius property*[1] *if* $\rho(A)$

[1]In the literature, there are two versions of the "Perron-Frobenius property", a strong one, corresponding to $\chi > 0$, and a weak one, corresponding to $\chi \geq 0$. In this paper we always consider the strong version.

is a simple positive eigenvalue of A *s.t.* $\rho(A) > |\lambda(A)|$ *for every* $\lambda(A) \in \mathrm{sp}(A)$, $\lambda(A) \neq \rho(A)$*, and* χ*, the right eigenvector relative to* $\rho(A)$*, is positive.*

The set of matrices which possess the Perron-Frobenius property will be denoted $\mathcal{PF}$, and it is known (see e.g. [22, Thms 8.2.8 and 8.4.4]) that positive matrices, as well as nonnegative and primitive matrices (i.e., matrices that are irreducible and have only one nonzero eigenvalue of maximum modulus), are part of this set. However, it has been shown (see [28]) that matrices having negative elements can also possess this property, provided that they are eventually positive.

Definition 2.2. *A matrix* $A \in \mathbb{R}^{n\times n}$ *is called eventually positive (denoted* $A \overset{\vee}{>} 0$*) if* $\exists\, k_0 \in \mathbb{N}$ *s.t.* $A^k > 0$ *for all* $k \geq k_0$*.*

Theorem 2.3. *[28, Thm 2.2] Let* $A \in \mathbb{R}^{n\times n}$*. The following are equivalent:*
1. *Both* $A, A^T \in \mathcal{PF}$*;*
2. $A \overset{\vee}{>} 0$*;*
3. $A^T \overset{\vee}{>} 0$.

Definition 2.4. *A matrix* $A \in \mathbb{R}^{n\times n}$ *is called eventually exponentially positive (EEP) if* $\exists\, t_0 \in \mathbb{N}$ *s.t.* $e^{At} > 0$ *for all* $t \geq t_0$*.*

Lemma 2.5. *[29, Thm 3.3] A matrix* $A \in \mathbb{R}^{n\times n}$ *is EEP if and only if* $\exists\, d \geq 0$ *s.t.* $A + dI \overset{\vee}{>} 0$*.*

3. Properties of signed Laplacian matrices.

The aim of this section is to summarize important properties of Laplacian matrices which will be useful in the following. Most of these results are from our previous works [3, 18, 19], hence they are reported here without proofs. First, Section 3.1 treats the unsigned Laplacians case; then, Section 3.2 considers the signed Laplacians case.

3.1. Unsigned graphs case.

When $\mathcal{G}(A)$ is a strongly connected unsigned digraph, it is well-known that its Laplacian L is a singular M-matrix, it is diagonally dominant, and it is marginally stable of corank 1. Its symmetric part in general need not be psd, but it is *Lyapunov diagonally semistable*, i.e., there exists a (unique) positive diagonal matrix $\Xi = \mathrm{diag}\,(\xi)$ $(\xi > 0)$ s.t. $\Xi L + L^T \Xi$ is psd. In particular, if L is weight balanced then its symmetric part is psd of corank 1.

Theorem 3.1 (Thm 2 and Coroll. 1 in [3]). *Let* $\mathcal{G}(A)$ *be an unsigned strongly connected digraph with Laplacian* L*. Then, the following hold:*
1. *Let* $\mathbb{1}$ *and* $\xi > 0$ *be the right and left eigenvectors of* L *relative to the eigenvalue* 0*. Then* ξ *is the unique (up to a scalar multiplication) positive vector for which the diagonal matrix* $\Xi = \mathrm{diag}\,(\xi)$ *is s.t.* $\Xi L + L^T \Xi$ *is psd. For it,* $\mathcal{N}(L^T\Xi) = \mathcal{N}(L) = \mathrm{span}(\mathbb{1})$ *and hence* $\Xi L + L^T \Xi$ *is of corank* 1*;*
2. $-L$ *is marginally stable of corank* 1.
3. *Assume that* L *is weight balanced. Then,* $L_s = \frac{L+L^T}{2}$ *is psd of corank* 1.

3.2. Signed graphs case.

Signed and unsigned Laplacians share some properties, such as having an eigenvalue in 0, but differ in others in subtle ways. For instance, while the Laplacian of an unsigned strongly connected digraph is always marginally stable, the same is not true in the signed case. Moreover, while it is well-known that in the unsigned case an irreducible Laplacian has a simple zero eigenvalue (i.e., $\mathrm{corank}(L) = 1$), this is not true in the signed case (see counterexamples in [18, 32]).

The following proposition summarizes these and other relevant observations.

Proposition 3.2. *Let* $\mathcal{G}(A)$ *be a signed digraph with signed Laplacian* L*. Then:*

(i) $0 \in \mathrm{sp}(L)$ *of right eigenvector* $\mathbb{1}$*;*
(ii) $-L$ *need not be marginally stable;*
(iii) $\mathrm{Re}[\lambda(L)] \geq 0$ *for all* $\lambda(L) \in \mathrm{sp}(L)$ *need not hold;*
(iv) L *need not be diagonally dominant;*
(v) L *irreducible (i.e.,* $\mathcal{G}(A)$ *strongly connected) need not imply* L *of corank* 1*.*

Concerning the converse of the last property, in both the signed and unsigned cases, $\mathrm{corank}(L) = 1$ implies that L has a rooted spanning tree. If in addition L is also weight balanced, then L is irreducible. Another sufficient condition for irreducibility is given by the EEP property.

LEMMA 3.3 (Lemma 5 in [19]). *Let* $\mathcal{G}(A)$ *be a signed digraph with signed Laplacian* L*.*

1. *If* L *is of corank* 1*, then* $\mathcal{G}(A)$ *has a rooted spanning tree.*
2. *If* $-L$ *is EEP or if* L *is weight balanced and of corank* 1*, then* L *is irreducible (and* $\mathcal{G}(A)$ *is strongly connected).*

In previous works, see [3, 19], we have investigated how to extend the results of Theorem 3.1 to the signed graph case. The main findings are summarized in Section 3.2.1 and Section 3.2.2 for the undirected and directed graphs case, respectively.

3.2.1. Signed undirected graphs case. The following theorem highlights the key role of the EEP property.

THEOREM 3.4 (Thm. 3 in [3]). *Let* $\mathcal{G}(A)$ *be a signed undirected graph with signed Laplacian* L*. Then, the following conditions are equivalent:*
(i) $-L$ *is EEP;*
(ii) $-L$ *is marginally stable of corank* 1*;*
(iii) L *is psd of corank* 1*.*

Remark 3.5. As per Lemma 3.3, it is redundant in Theorem 3.4 (and in the following theorems) to add the assumption that the signed graph $\mathcal{G}(A)$ must be strongly connected.

3.2.2. Signed directed graphs case. When the signed graph $\mathcal{G}(A)$ is directed, the conditions of Theorem 3.4 are no longer equivalent: EEP of the signed Laplacian is a sufficient but not necessary condition for its marginal stability. Moreover, even if $-L$ is EEP (or marginally stable of corank 1) its symmetric part may not be psd. Theorem 3.6 extends the results of Theorem 3.4 to signed directed graphs, and shows that for digraphs that are weight balanced, EEP and marginal stability (of corank 1) of the signed Laplacian are equivalent properties. Additionally, by further restring to digraphs whose Laplacian is a normal matrix, stability of the symmetric part of the Laplacian can be guaranteed.

THEOREM 3.6 (Thm. 4, Cor. 1, and Cor. 2 in [19]). *Let* $\mathcal{G}(A)$ *be a signed directed graph with signed Laplacian* L*. Consider the following conditions:*
(i) $-L$ *is EEP;*
(ii) $-L$ *is marginally stable of corank* 1*;*
(iii) $L_s = \frac{L+L^T}{2}$ *is psd of corank* 1*.*

1. *If* L *satisfies (i), then* L *satisfies (ii). Viceversa, if* L *satisfies (ii), then there exists a scalar* $d \geq 0$ *such that* $dI - L \in \mathcal{PF}$*.*
2. *If* L *is s.t.* L_s *satisfies (iii), then* L *satisfies (i) and (ii), but not viceversa.*
3. *If* L *is weight balanced, then the conditions (i) and (ii) are equivalent, and both are implied by (iii), but not viceversa.*

4. *If L is normal, then (i), (ii), and (iii) are equivalent.*

Condition (iii) of Theorem 3.6 corresponds obviously to $-L_s$ EEP, see Theorem 3.4.

4. Pseudoinverse of signed Laplacians. This section contains the main results of the paper. Consider a signed digraph $\mathcal{G}(A)$ with signed Laplacian L. We start by listing a few useful properties of L and $L^\dagger$. Assume that L is weight balanced of corank 1. Then L is a range symmetric matrix with $\mathcal{N}(L) = \mathcal{N}(L^T) = \text{span}(\mathbb{1})$. Let $\Pi = I - J$, where $J = \frac{\mathbb{1}\mathbb{1}^T}{n}$, denote the projection of $\mathbb{R}^n$ onto $\mathcal{R}(L) = \mathcal{R}(L^T) = \mathbb{1}^\perp$, i.e., the subspace of $\mathbb{R}^n$ orthogonal to $\mathbb{1}$. A few properties of L follow straightforwardly.

LEMMA 4.1. *The matrix $J = \frac{\mathbb{1}\mathbb{1}^T}{n}$ has the following properties:*
1. $J = \lim_{t\to\infty} e^{-Lt} = \lim_{t\to\infty} e^{-L^T t}$*;*
2. $J^k = J \ \forall k \in \mathbb{N}$ *which implies that* $(I - J)^k = (I - J) \ \forall k \in \mathbb{N}$*;*
3. $JL = LJ = 0$ *which implies that* $e^{-(L+J)} = e^{-L}e^{-J}$ *and* $Je^{-L} = e^{-L}J = J$*;*
4. $e^{-Jt} = I - J + Je^{-t}$ *which implies that* $Je^{-Jt} = e^{-Jt}J = Je^{-t}$*.*

The Laplacian pseudoinverse $L^\dagger$ of L satisfies the following properties.

LEMMA 4.2. *If L is weight balanced and of corank 1, then $L^\dagger$ is weight balanced and of corank 1. For it*

$$LL^\dagger = L^\dagger L = \Pi \tag{4.1a}$$

$$L^\dagger \mathbb{1} = (L^\dagger)^T \mathbb{1} = 0 \tag{4.1b}$$

$$L^\dagger \Pi = \Pi L^\dagger = L^\dagger \tag{4.1c}$$

$$L^\dagger = (L + \gamma J)^{-1} - \frac{1}{\gamma} J \quad \forall \gamma \neq 0. \tag{4.1d}$$

Furthermore, if L is normal then $L^\dagger$ is normal.

Proof in Appendix A.

Remark 4.3. Lemmas 4.1 and 4.2 hold also for any unsigned Laplacian matrix L.

In the next two sections, Sections 4.1 and 4.2, our main results on the pseudoinverses of Laplacian matrices are presented, in the unsigned and signed graph case, respectively.

4.1. Unsigned graphs case. The class of unsigned Laplacians is not closed with respect to pseudoinversion. In fact, as e.g. (1.1)-(1.2) show, the pseudoinverse of an unsigned L is in general a signed Laplacian. The following theorem states this fact, and shows that all other properties of relevance for a Laplacian (Theorem 3.1) are nevertheless respected. It also shows that for non-symmetric L there is more than one way to define the symmetric part for the pseudoinverse.

THEOREM 4.4. *Let $\mathcal{G}(A)$ be an unsigned strongly connected digraph with Laplacian L, and assume that L is weight balanced. Let $L^\dagger$ be the (weight balanced) pseudoinverse of L. Then:*
(i) $-L^\dagger$ *is EEP;*
(ii) $-L^\dagger$ *is marginally stable of corank* 1*;*
(iii) $(L^\dagger)_s = \frac{L^\dagger + (L^\dagger)^T}{2}$ *is psd of corank* 1*;*
(iv) $(L_s)^\dagger = \left(\frac{L+L^T}{2}\right)^\dagger$ *is psd of corank* 1*.*

Proof in Appendix B.

Example 4.5. The pseudoinverse of the unsigned (symmetric) Laplacian matrix (1.1) is given in (1.2). Since the element in position (1,2) is positive, $L^\dagger$ is not a

Z-matrix and hence it is not an unsigned Laplacian matrix, but it is rather a signed Laplacian matrix. Moreover, $\mathrm{sp}(L^\dagger) = \{0, 0.64, 2.26\}$, that is, $-L^\dagger$ is marginally stable. Combined with property (4.1b) in Lemma 4.2, $L^\dagger$ is also EEP.

Remark 4.6. For digraphs, in general $(L_s)^\dagger \neq (L^\dagger)_s$, meaning that the operations of taking the symmetric part and of taking the pseudoinverse do not commute, i.e., the following diagram does not commute

$$\begin{array}{ccc} L & \xrightarrow{\text{pseudoinv.}} & L^\dagger \\ \text{symm.}\downarrow & & \downarrow\text{symm.} \\ L_s & \xrightarrow{\text{pseudoinv.}} & (L_s)^\dagger \neq (L^\dagger)_s \end{array} \tag{4.2}$$

See Example 4.7 for a counterexample.

Example 4.7. Consider the following unsigned weight balanced Laplacian matrix L, whose (weight balanced) pseudoinverse is given by $L^\dagger$:

$$L = \begin{bmatrix} 0.49 & -0.49 & 0 & 0 \\ -0.15 & 0.59 & -0.07 & -0.37 \\ 0 & 0 & 0.49 & -0.49 \\ -0.34 & -0.1 & -0.42 & 0.86 \end{bmatrix}, \quad L^\dagger = \begin{bmatrix} 1.24 & 0.49 & -1.02 & -0.66 \\ -0.31 & 0.99 & -0.5 & -0.15 \\ -0.72 & -1 & 1.51 & 0.14 \\ -0.21 & -0.48 & 0.01 & 0.67 \end{bmatrix}.$$

It is $\mathrm{sp}(L) = \{0, 0.42, 0.98, 1.03\}$, $\mathrm{sp}(L_s) = \{0, 0.34, 0.86, 1.22\}$, $\mathrm{sp}((L_s)^\dagger) = \{0, 0.82, 1.16, 2.90\}$, $\mathrm{sp}(L^\dagger) = \{0, 0.97, 1.02, 2.40\}$, and $\mathrm{sp}((L^\dagger)_s) = \{0, 0.77, 1.02, 2.59\}$, that is, $-L, -L^\dagger$ are marginally stable of corank 1 and $L_s, (L_s)^\dagger, (L^\dagger)_s$ are psd of corank 1.

4.2. Signed graphs case. As shown in Theorems 3.4 and 3.6, the conditions $-L$ EEP and $-L$ marginally stable of corank 1 are equivalent, meaning that the class of weight balanced signed Laplacian matrices which are EEP is closed with respect to stability. Our main aim in this Section is to show that this class is closed also with respect to pseudoinversion.

4.2.1. Signed undirected graphs case. For the class of symmetric Laplacian matrices which are EEP, Theorem 4.8 extends the results of Theorem 3.4 and shows closure with respect to pseudoinversion. Furthermore, Theorem 4.9 shows that this class is closed also under Kron reduction, meaning that the Kron reduced matrix of an EEP signed Laplacian is also a signed Laplacian which is EEP.

THEOREM 4.8. *Let $\mathcal{G}(A)$ be a signed undirected graph with signed Laplacian L. Let $L^\dagger$ be the pseudoinverse of L. Then, the following conditions are equivalent:*
(i) $-L$ is EEP;
(ii) $L^\dagger$ is psd of corank 1;
(iii) $-L^\dagger$ is EEP.

Proof in Appendix C.

THEOREM 4.9. *Let $\mathcal{G}(A)$ be a signed undirected graph with signed Laplacian L. Let α (with $\mathrm{card}(\alpha) \in [2, n-1]$) and $\beta = \{1, \dots, n\} \setminus \alpha$ be a partition of the node set $\mathcal{V}$. Let $\mathcal{G}_r$ be the signed undirected graph obtained by applying the Kron reduction on $\mathcal{G}$, and let $L_r = L/L[\beta]$ be its Laplacian. Consider the following conditions:*
(i) $-L$ is EEP;
(ii) L_r is psd of corank 1;
(iii) $-L_r$ is EEP.

If L satisfies (i), then L_r satisfies (ii) and (iii).

Furthermore, if $\mathcal{G}(A)$ is connected, α is the set of nodes incident to negatively weighted edges, and $\beta = \{1, \dots, n\} \backslash \alpha$, then the conditions (i), (ii), (iii) are equivalent.

Proof in Appendix C.

Note that if the set α of Theorem 4.9 does not correspond to the set of nodes incident to negatively weighted edges then, even if L_r is psd of corank 1 and L is irreducible, $-L$ need not be EEP.

The results of this section can be summarized in the following corollary.

COROLLARY 4.10. *The class of EEP symmetric Laplacian matrices is closed under the pseudoinverse operation, under the operation of Kron reduction, and with respect to stability.*

We conclude this section by observing that the class of EEP symmetric Laplacian matrices described in Corollary 4.10 is also closed with respect to (positive) summation.

LEMMA 4.11. *Consider two undirected signed graphs $\mathcal{G}(A_i)$ with signed Laplacian L_i, $i = 1, 2$. If $-L_i$, $i = 1, 2$, is EEP, then the matrix $L = k_1 L_1 + k_2 L_2$, where k_1, k_2 are positive scalars, is itself a signed Laplacian and $-L$ is EEP.*

Proof in Appendix C.

4.2.2. Signed directed graphs case. The results of Theorem 3.6 hold also for the Laplacian pseudoinverse, as shown in Theorem 4.12, which extends the results of Theorems 4.4 and 4.8 to signed directed graphs.

THEOREM 4.12. *Let $\mathcal{G}(A)$ be a signed directed graph with signed Laplacian L, and assume that L is weight balanced. Let $L^\dagger$ be pseudoinverse of L. Then, the following conditions are equivalent:*

(i) $-L$ is EEP;

(ii) $-L^\dagger$ is marginally stable of corank 1;

(iii) $-L^\dagger$ is EEP.

Furthermore, consider the following statements:

(iv) $(L^\dagger)_s = \frac{L^\dagger + (L^\dagger)^T}{2}$ is psd of corank 1;

(v) $(L_s)^\dagger = \left(\frac{L+L^T}{2}\right)^\dagger$ is psd of corank 1.

If L is normal, then (i)÷(v) are equivalent.

Proof in Appendix D.

Remark 4.13. Even in the case of a normal Laplacian L, the operations of pseudoinverse and of symmetrization do not commute, i.e., $(L^\dagger)_s \neq (L_s)^\dagger$. Proof in Appendix D.

In [41] the authors introduce a new notion of "generalized inverse" of the Laplacian matrix for unsigned digraphs. They observe that, since the Laplacian L of an unsigned graph is marginally stable of corank 1, then its projection on $\mathbb{1}^\perp$, denoted $\bar{L} = QLQ^T$ where the rows of $Q \in \mathbb{R}^{n-1 \times n}$ form an orthonormal basis for $\mathbb{1}^\perp$, is Hurwitz stable. Therefore, there exists a unique pd matrix S which solves the Lyapunov equation $\bar{L}S + S\bar{L}^T = I_{n-1}$. They proceed to define the "generalized inverse" as $X = 2Q^T SQ$, which has the property of being a positive semidefinite matrix. The reasoning of [41] is valid also for signed digraphs, provided that L is normal. In particular, in the next lemma we show that, if L is normal and $-L$ is EEP, X is equivalent to $(L_s)^\dagger$.

LEMMA 4.14. *Let $\mathcal{G}(A)$ be a signed digraph with Laplacian L, and assume that L is normal and $-L$ is EEP. Then, $(L_s)^\dagger = X$, where*

$$X = 2Q^T SQ, \quad \bar{L}S + S\bar{L}^T = I_{n-1}, \quad \bar{L} = QLQ^T. \tag{4.3}$$

Proof in Appendix D.

The results of this section can be summarized in the following corollary.

COROLLARY 4.15. *The class of EEP weight balanced Laplacian matrices is closed under the pseudoinversion operation, and with respect to stability.*

The class of EEP normal Laplacian matrices is closed under any combination of pseudoinverse and symmetrization.

Finally, notice that the class of EEP weight balanced Laplacian matrices is not a cone and, for instance, Lemma 4.11 does not hold in the directed case. However, it is possible to show that this class is star-shaped, meaning that it is path-connected [23] (see also [5, Def. 5.4] for a definition of star-shaped set).

LEMMA 4.16. *The class of EEP weight balanced Laplacian matrices is star-shaped with respect to the star center $\Pi = I - \frac{\mathbb{1}\mathbb{1}^T}{n}$, i.e., $L_\alpha := \alpha L + (1-\alpha)\Pi$, $\alpha \in [0,1]$, is a weight balanced signed Laplacian, and its negation $-L_\alpha$ is EEP.*

Proof in Appendix D.

4.3. Properties of signed Laplacians and their pseudoinverses: a summary. This section summarizes the inclusion properties of the classes of signed Laplacian matrices considered in this work. Let L be a signed Laplacian and $L^\dagger$ its pseudoinverse. It holds that:

$$\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \supset \mathcal{C}_4 \supset \mathcal{C}_5 \tag{4.4}$$

where $\mathcal{C}_1 = \{L\colon -L \text{ is marginally stable (of corank 1)}\}$, $\mathcal{C}_2 = \{L\colon -L \text{ is EEP}\}$, $\mathcal{C}_3 = \{L\colon -L \text{ is EEP}, L\mathbb{1} = L^T\mathbb{1}\}$, $\mathcal{C}_4 = \{L\colon -L_s \text{ is EEP}\}$, and $\mathcal{C}_5 = \{L\colon -L \text{ is EEP}, L \text{ is normal}\}$. From Corollaries 4.10 and 4.15, we have:

- the sets $\mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$ are closed w.r.t. pseudoinversion and marginal stability;
- the set $\mathcal{C}_5$ is closed under any combination of pseudoinversion and symmetrization.

Consequently we could also have written: $\mathcal{C}_3 = \{L^\dagger\colon -L^\dagger \text{ is EEP}, L^\dagger\mathbb{1} = (L^\dagger)^T\mathbb{1}\}$, $\mathcal{C}_4 = \left\{L^\dagger\colon -(L^\dagger)_s \text{ is EEP}\right\}$, and $\mathcal{C}_5 = \{L^\dagger\colon -L^\dagger \text{ is EEP}, L^\dagger \text{ is normal}\}$.

Using counterexamples, we can show that the inequalities in (4.4) are strict.

Example 4.17. In this example we show that the inequalities in (4.4) are strict.

- $\mathcal{C}_2 \subsetneq \mathcal{C}_1$. Consider the following signed Laplacian matrix

$$L = \begin{bmatrix} -0.4 & 0.7 & 0 & -0.3 \\ -1.4 & 1.6 & 0.2 & -0.4 \\ -0.7 & 0 & 2.8 & -2.1 \\ 0 & 0 & -1.3 & 1.3 \end{bmatrix}.$$

  It is $\text{sp}(L) = \{0, 0.73 \pm 0.12i, 3.83\}$, i.e., $-L$ marginally stable, but the left eigenvector associated to 0, $[0.78 \ -0.34 \ 0.24 \ 0.46]^T$, is not positive, i.e., $-L$ is not EEP.
- $\mathcal{C}_3 \subsetneq \mathcal{C}_2$. Consider the following signed Laplacian matrix

$$L = \begin{bmatrix} 0.73 & 0 & -0.73 & 0 \\ 0 & 1.02 & -0.4 & -0.62 \\ 0 & -0.07 & 0.7 & -0.63 \\ -0.63 & 0.05 & 0 & 0.57 \end{bmatrix}.$$

It is $\mathrm{sp}(L) = \{0, 0.97 \pm 0.58i, 1.08\}$, i.e., $-L$ marginally stable, and the left eigenvector associated to 0, $[0.54\ \ 0.01\ \ 0.57\ \ 0.63]^T$, is positive, i.e., $-L$ is EEP: for $d > 0.6572$, $B = dI - L \overset{\vee}{>} 0$. However, $L\mathbb{1} \neq L^T\mathbb{1}$, i.e., L is not weight balanced.

- $\mathcal{C}_4 \subsetneq \mathcal{C}_3$. Consider the following signed Laplacian matrix

$$L = \begin{bmatrix} 0.15 & 0 & 0 & -0.15 \\ -0.23 & 0.15 & 0.15 & -0.07 \\ 0.01 & -0.12 & -0.03 & 0.14 \\ 0.07 & -0.03 & -0.12 & 0.08 \end{bmatrix}.$$

It is $\mathrm{sp}(L) = \{0, 0.0901 \pm 0.199i, 0.169\}$, i.e., $-L$ is marginally stable of corank 1. Moreover, $L\mathbb{1} = L^T\mathbb{1} = 0$ and, for $d > 0.2647$, $B = dI - L \overset{\vee}{>} 0$. However, $\mathrm{sp}(L_s) = \{-0.0402, 0, 0.1248, 0.2655\}$, i.e., L_s is not psd.

- $\mathcal{C}_5 \subsetneq \mathcal{C}_4$. Consider the following signed Laplacian matrix

$$L = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

It is $\mathrm{sp}(L) = \{0, 1.5 \pm 1.323i, 2\}$, i.e., $-L$ is marginally stable of corank 1, and $\mathrm{sp}(L_s) = \{0, 0.7192, 1.5, 2.7808\}$, i.e., L_s is psd of corank 1. Moreover, $L\mathbb{1} = L^T\mathbb{1} = 0$, but $LL^T \neq L^T L$, that is, L is not normal.

5. Application to effective resistance.

A resistive electrical network can be represented as a graph $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E}, A)$ where each weight a_{ij} represents the inverse of the resistance between the nodes i and j (i.e., the conductance of the transmission): $a_{ij} = \frac{1}{r_{ij}}$, see [24, 20], and [15] for an overview. The notion of effective resistance between a pair of nodes (see e.g. [15]) is related to the pseudoinverse of the Laplacian associated to the electrical network. When the network is connected, undirected and nonnegative, its Laplacian (and its pseudoinverse) is known to be psd of corank 1, which means that the effective resistance between two nodes is well-defined (see e.g. [20] for its properties). Extensions to signed graphs and negative resistances have been investigated in [43, 11, 44, 9, 10], where positive semidefiniteness of the Laplacian is expressed in terms of effective resistance.

In what follows we make use of both $(L^\dagger)_s$ and $(L_s)^\dagger$ to extend the notion of effective resistance to directed signed networks whose Laplacian L is a normal matrix and $-L$ is EEP. As already observed in Remarks 4.6 and 4.13, when the network is directed $(L^\dagger)_s$ and $(L_s)^\dagger$ are no longer equivalent, which motivates us to propose a definition that encompasses both notions. As explained more in details below in Section 5.1, one of the two notions is novel, while the other extends an available definition to the signed graph case.

Definition 5.1. *The effective resistance between two nodes $i, j \in \{1, \dots, n\}$ of a signed digraph whose corresponding Laplacian L is normal and s.t. $-L$ is EEP, is given by*

$$\begin{aligned} R_{ij}(X) &= [X]_{ii} + [X]_{jj} - [X]_{ij} - [X]_{ji} \\ &= (e_i - e_j)^T X (e_i - e_j), \quad X \in \{(L_s)^\dagger, (L^\dagger)_s\} \end{aligned} \tag{5.1}$$

i.e., $X = [X]_{ij}$ is either given by the pseudoinverse of the symmetrization of the Laplacian $(L_s)^\dagger$, or by the symmetrization of the Laplacian pseudoinverse $(L^\dagger)_s$. The effective resistance matrix $R(X) = [R_{ij}(X)]$ is defined as

$$R(X) = D_X \mathbb{1}\mathbb{1}^T + \mathbb{1}\mathbb{1}^T D_X - 2X, \quad X \in \{(L_s)^\dagger, (L^\dagger)_s\} \tag{5.2}$$

where $D_X = \operatorname{diag}([X]_{11}, \ldots, [X]_{nn})$ is a diagonal matrix whose elements are the diagonal elements of X. The total effective resistance is defined as

$$R_{\rm tot}(X) = \frac{1}{2}\mathbb{1}^T R(X)\mathbb{1}, \quad X \in \{(L_s)^\dagger, (L^\dagger)_s\}. \tag{5.3}$$

As its counterpart for undirected graphs (see [24, 20, 15]), for both $X \in \{(L_s)^\dagger, (L^\dagger)_s\}$ the effective resistance (5.1) is still nonnegative and symmetric. Its square root is a metric, and the effective resistance matrix (5.2) is a Euclidean distance matrix, i.e., it has nonnegative elements, zero diagonal elements, and it is negative semidefinite on $\mathbb{1}^\perp$ [20], see the following lemma.

LEMMA 5.2. *The square root of the effective resistance* (5.1) *between two nodes $i, j \in \{1, \ldots, n\}$ of a signed digraph with normal Laplacian L is a metric: it is nonnegative, symmetric and it satisfies the triangle inequality. The effective resistance matrix* (5.2) *is a Euclidean distance matrix.*

Proof in Appendix E. The last part of the proof follows [20, Section 2.8] and is here reported for completeness.

Remark 5.3. For digraphs, the main difference between $(L^\dagger)_s$ and $(L_s)^\dagger$ is that in the first the pseudoinverse respects the physical asymmetric nature of the problem, while in the latter any asymmetry is lost when taking the pseudoinverse. This affects the two values of effective resistance $R((L^\dagger)_s)$ and $R((L_s)^\dagger)$. In particular, from (4.2) we have that $R((L_s)^\dagger) \neq R((L^\dagger)_s)$, as the following lemma states.

LEMMA 5.4. *Let $\mathcal{G}(A)$ be a signed graph with signed Laplacian L, and assume that L is normal and $-L$ is EEP.*

(i) The effective resistances $R_{ij}((L_s)^\dagger)$ and $R_{ij}((L^\dagger)_s)$, defined in (5.1), *satisfy*

$$R_{ij}((L^\dagger)_s) \le R_{ij}((L_s)^\dagger) \quad i, j = 1, \ldots, n.$$

(ii) The total effective resistances $R_{\rm tot}((L_s)^\dagger)$ and $R_{\rm tot}((L^\dagger)_s)$, defined in (5.3), *satisfy*

$$R_{\rm tot}((L_s)^\dagger) = n\sum_{i=2}^{n} \frac{1}{\operatorname{Re}[\lambda_i(L)]}, \quad R_{\rm tot}((L^\dagger)_s) = n\sum_{i=2}^{n} \operatorname{Re}[\lambda_i(L^\dagger)]$$

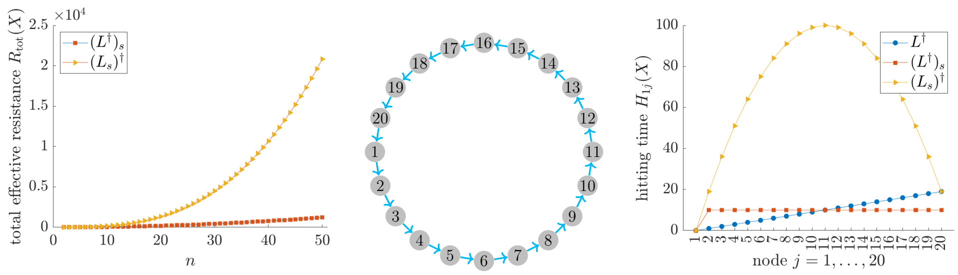
$$R_{\rm tot}((L^\dagger)_s) \le R_{\rm tot}((L_s)^\dagger).$$

Proof in Appendix E.

Remark 5.5. $R_{ij}(X)$ of eq. (5.1) is a quadratic form generated by the matrix X, i.e., only the symmetric part of X matters: $R_{ij}(X) = R_{ij}\left(\frac{X+X^T}{2}\right)$. When $X = (L^\dagger)_s$ this has a twofold consequence. First, it is

$$R_{ij}((L^\dagger)_s) = R_{ij}(L^\dagger) \tag{5.4}$$

i.e., the effective resistance can be built directly from $L^\dagger$ without any symmetrization on the Laplacian. Second, for signed graphs, to ensure that $R_{ij}(L^\dagger)$ is well-defined



(a) (b) (c)

Figure 1: Example 5.7. (a): total effective resistance $R_{\rm tot}(X)$, with $X \in \{(L^\dagger)_s, (L_s)^\dagger\}$, for a sequence of cycle unsigned digraphs with increasing number of nodes, $n = 2, \ldots, 50$. (b): cycle unsigned digraph with $n = 20$. (c): hitting times $H_{ij}(X)$ from node $i = 1$ to node $j = 1, \ldots, 20$, with $X \in \{L^\dagger, (L^\dagger)_s, (L_s)^\dagger\}$.

(i.e., $R_{ij}(L^\dagger) \geq 0$ for all i, j), EEP of $-L^\dagger$ is not sufficient. From Theorem 4.12, a normality assumption on the Laplacian must be added in Definition 5.1. Notice that on signed digraphs the same assumption is needed also for the other version of effective resistance given in Definition 5.1, in order to guarantee that $R((L_s)^\dagger) \geq 0$ for all i, j, see Theorem 3.6.

Remark 5.6. Definition 5.1 becomes less restrictive in the case of unsigned digraphs. In that case, it is sufficient to assume that the Laplacian is weight balanced and irreducible since, applying Theorem 4.4, it holds that both $(L_s)^\dagger$ and $(L^\dagger)_s$ are psd of corank 1.

Example 5.7. Let $\mathcal{G}(A)$ be a nonnegative, unweighted, directed, cycle graph (see Fig. 1b), whose Laplacian L is a normal matrix with eigenvalues $1+e^{i\theta_k}$, with $\theta_k = \pi\big(1 - \frac{2k}{n}\big)$, for all $k = 0, \ldots, n-1$. Then, $R_{\rm tot}((L_s)^\dagger) = \frac{n(n^2-1)}{6}$ (see e.g. [39]), $R_{\rm tot}((L^\dagger)_s) = n\sum_{k=2}^n {\rm Re}[\frac{1}{\lambda_k(L)}] = n\sum_{k=2}^n \frac{1+\cos\theta_k}{(1+\cos\theta_k)^2+\sin^2\theta_k} = n\sum_{k=2}^n \frac{1}{2} = \frac{n(n-1)}{2}$, and we obtain $R_{\rm tot}((L^\dagger)_s) \leq R_{\rm tot}((L_s)^\dagger)$ for all $n \geq 2$, see Fig. 1a.

The two notions of effective resistance in (5.1) differ also w.r.t. Rayleigh's monotonicity law. While $R((L_s)^\dagger)$ obeys it (see Lemma 5.8), $R((L^\dagger)_s)$ does not (see counterexample 5.9).

LEMMA 5.8. *Consider two signed digraphs $\mathcal{G}(A_i)$ with signed Laplacian L_i, $i = 1, 2$. Assume that L_i is normal and that $-L_i$ is EEP, $i = 1, 2$. If $A_1 \geq A_2$ (componentwise) then $R_{\rm tot}((L_{1s})^\dagger) \leq R_{\rm tot}((L_{2s})^\dagger)$, where $R_{\rm tot}((L_{is})^\dagger)$ $(i = 1, 2)$ is the total effective resistance associated with $\mathcal{G}(A_i)$.*

Proof in Appendix E.

Example 5.9. Consider the following signed Laplacian matrices

$$L_1 = \begin{bmatrix} 0.34 & -0.23 & 0.18 & -0.29 \\ -0.23 & 0.49 & -0.05 & -0.21 \\ -0.29 & -0.21 & 0.26 & 0.24 \\ 0.18 & -0.05 & -0.39 & 0.26 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.16 & -0.19 & 0.25 & -0.22 \\ -0.19 & 0.34 & 0 & -0.15 \\ -0.22 & -0.15 & 0.07 & 0.3 \\ 0.25 & 0 & -0.32 & 0.07 \end{bmatrix}.$$

Both L_1 and L_2 are normal and it is ${\rm sp}(L_1) = \{0, 0.33 \pm 0.50i, 0.68\}$ and ${\rm sp}(L_2) =$

$\{0, 0.49, 0.77 \pm 0.46i\}$, i.e., $-L_1, -L_2$ are marginally stable of corank 1. Then, $-L_1$ and $-L_2$ are EEP. The corresponding adjacency matrices A_1, A_2 satisfy $A_1 \geq A_2$. The total effective resistances associated with $\mathcal{G}(A_1), \mathcal{G}(A_2)$ satisfy:

$$R_{\rm tot}((L_{1s})^\dagger) = 29.83 \leq 111.89 = R_{\rm tot}((L_{2s})^\dagger)$$
$$R_{\rm tot}((L_1^\dagger)_s) = 13.92 \geq 11.01 = R_{\rm tot}((L_2^\dagger)_s)$$

Only the effective resistance calculated according to $(L_s)^\dagger$ obeys Rayleigh's monotonicity law.

5.1. Comparison with other notions of effective resistance. Of the two notions in Definition 5.1, one, $R((L^\dagger)_s)$, is novel and proposed here for the first time. The other, $R((L_s)^\dagger)$, has already been used in the literature, but not for signed digraphs. In [39, 41] the authors introduce a notion of effective resistance for strongly connected unsigned digraphs. As shown in Lemma 4.14, their effective distance is based on the pseudoinverse of the symmetrization $(L_s)^\dagger$, and can be extended to signed digraphs. It corresponds to $R((L_s)^\dagger)$ computed in (5.2) whenever $(L_s)^\dagger$ can be computed. Formally the definition of [41] can be stated as

$$R(X) = D_X \mathbb{1}\mathbb{1}^T + \mathbb{1}\mathbb{1}^T D_X - 2X, \text{ where } X \text{ satisfies (4.3)} \tag{5.5}$$

Comparing our $R((L_s)^\dagger)$ to (5.5) we have:
- The definition (5.5) was developed for unsigned strongly connected digraphs and does not require L to be normal, nor weight balanced;
- Our $R((L_s)^\dagger)$ is valid for signed graphs for which L is normal and $-L$ EEP.

The notion of effective resistance (5.5) has been considered e.g. in [17], where the author proposes a symmetrization of digraphs which preserves pairwise effective resistances.

6. Further applications and extensions: an outlook.

In this section we outline a few possible further applications of our signed Laplacian pseudoinverse to other contexts.

6.1. Effective vs equivalent conductance. A concept often associated to effective resistance is that of effective conductance C, defined as the Hadamard inverse of R (see e.g. [24]): $C_{ij} = \frac{1}{R_{ij}}$. For Laplacians that are normal and EEP, we can use our notions of pseudoinverse to extend it to signed digraphs in the intuitive way, as

$$C_{ij}(X) = \begin{cases} \frac{1}{(e_i - e_j)^T X (e_i - e_j)}, & i \neq j, \quad X \in \{(L_s)^\dagger, (L^\dagger)_s\} \\ 0, & i = j \end{cases} \tag{6.1}$$

However, an alternative definition is also possible, reflecting the fact that such Laplacians and their pseudoinverses share the same properties (Corollary 4.15). To avoid ambiguity in the terminology, we refer to this new concept as *equivalent conductance.*

DEFINITION 6.1. *The equivalent conductance between two nodes $i, j \in \{1, \dots, n\}$ of a signed digraph whose corresponding Laplacian L is normal and $-L$ is EEP, is given by*

$$\tilde{C}_{ij}(X) = (e_i - e_j)^T X (e_i - e_j), \quad X \in \{L_s, ((L^\dagger)_s)^\dagger\} \tag{6.2}$$

where $X = [X]_{ij}$ is either given by the symmetrization of the Laplacian L_s, or by the pseudoinverse of the symmetrization of the Laplacian pseudoinverse $((L^\dagger)_s)^\dagger$. The *equivalent conductance matrix* $\tilde{C}(X) = [\tilde{C}_{ij}(X)]$ *is defined as*

$$\tilde{C}(X) = D_X \mathbb{1}\mathbb{1}^T + \mathbb{1}\mathbb{1}^T D_X - 2X, \quad X \in \{L_s, ((L^\dagger)_s)^\dagger\} \tag{6.3}$$

where $D_X = \operatorname{diag}([X]_{11}, \dots, [X]_{nn})$*. The total equivalent conductance is defined as*

$$\tilde{C}_{\text{tot}}(X) = \frac{1}{2}\mathbb{1}^T \tilde{C}(X)\mathbb{1}, \quad X \in \{L_s, ((L^\dagger)_s)^\dagger\}. \tag{6.4}$$

Obviously, as in (5.4), $\tilde{C}_{ij}(L_s) = \tilde{C}_{ij}(L)$. The equivalent conductance shares the properties of the effective resistance listed in Lemma 5.2 and Lemma 5.4:

- The square root of the equivalent conductance matrix $\tilde{C}$ in (6.3) is a metric, and $\tilde{C}$ is a Euclidean distance matrix;
- The equivalent conductances $\tilde{C}_{ij}$ (6.2) satisfy: $\tilde{C}_{ij}(L_s) \leq \tilde{C}_{ij}(((L^\dagger)_s)^\dagger)$, for all $i, j = 1, \dots, n$. The total equivalent conductances $\tilde{C}_{\text{tot}}$ (6.4) satisfy: $\tilde{C}_{\text{tot}}(L_s) = n \cdot \sum_{i=2}^n \text{Re}[\lambda_i(L)]$, $\tilde{C}_{\text{tot}}(((L^\dagger)_s)^\dagger) = n \cdot \sum_{i=2}^n \frac{1}{\text{Re}[\lambda_i(L^\dagger)]}$, and $\tilde{C}_{\text{tot}}(L_s) \leq \tilde{C}_{\text{tot}}(((L^\dagger)_s)^\dagger)$.

Instead, the effective conductance C in (6.1) in general does not share all the properties of the effective resistance, as shown in the following example.

Example 6.2. Consider the following signed Laplacian matrix

$$L = \begin{bmatrix} 5.94 & -2.61 & 1.79 & 1.21 & -6.32 \\ -2.61 & 7.76 & -0.82 & -1 & -3.33 \\ 1.79 & -0.82 & 0.65 & 0.36 & -1.97 \\ -6.32 & -3.33 & -1.97 & 7.67 & 3.95 \\ 1.21 & -1 & 0.36 & -8.24 & 7.67 \end{bmatrix},$$

which is normal and such that $-L$ is EEP.

To show that the effective conductance is *not* a Euclidean distance matrix, we show that there exists $z \perp \mathbb{1}$ such that $z^T C(X) z \geq 0$, $X \in \{(L_s)^\dagger, (L^\dagger)_s\}$. With $z = [-2.6\ 0.7\ 0.5\ 0.4\ 1]^T \in \text{span}(\mathbb{1}^\perp)$ it is $z^T C((L_s)^\dagger) z = 3.4170$, $z^T C((L^\dagger)_s) z = 8.3626$.

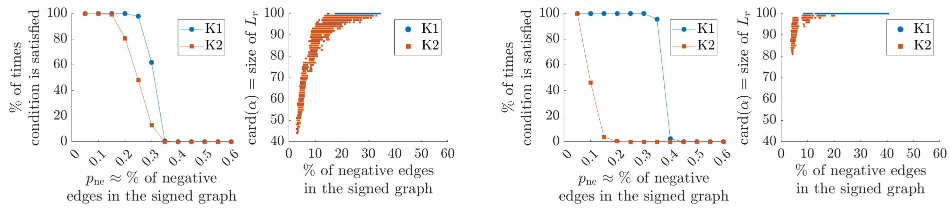
To show that the square root of the effective conductance is *not* a metric, we show that the triangle inequality does not hold. Let $i = 1, k = 3, j = 4$; it is

$$\sqrt{C_{13}((L_s)^\dagger)} + \sqrt{C_{34}((L_s)^\dagger)} = 0.5819 \leq 0.9689 = \sqrt{C_{14}((L_s)^\dagger)},$$
$$\sqrt{C_{13}((L^\dagger)_s)} + \sqrt{C_{34}((L^\dagger)_s)} = 0.5827 \leq 1.0065 = \sqrt{C_{14}((L^\dagger)_s)}.$$

6.2. Kron reduction vs EEP for undirected signed graphs. As Theorem 4.9 shows, for undirected graphs the Kron reduction procedure can be extended to signed graphs. Similarly to the unsigned graph case (see e.g. [15, Proposition 5.8]), one of the features of Kron reduction on signed graphs is that the effective resistance is invariant under Kron reduction, as shown in the following lemma.

LEMMA 6.3. *Let* $\mathcal{G}(A)$ *be a signed undirected graph with signed Laplacian* L*, and assume that* $-L$ *is EEP. Let* α *(with* $\text{card}(\alpha) \in [2, n-1]$*) and* $\beta = \{1, \dots, n\} \setminus \alpha$ *be a partition of the node set* $\mathcal{V}$*. Let* $\mathcal{G}_r$ *be the signed undirected graph obtained by applying the Kron reduction on* $\mathcal{G}$*, and let* $L_r = L/L[\beta]$ *be its Laplacian. Then, the effective resistance* (5.1) *between two nodes* $i, j \in \alpha$ *can be equivalently computed as:*

$$R_{ij}(L^\dagger) = (e_i - e_j)^T L^\dagger (e_i - e_j) = (e_i[\alpha] - e_j[\alpha])^T (L_r)^\dagger (e_i[\alpha] - e_j[\alpha]) := R_{ij}((L_r)^\dagger).$$



(a) (b)

Figure 2: Example 6.4. Conditions "$-L$ is EEP" (K1) and "L_r is psd and $L_r \neq L$" (K2) (left panels) and corresponding size of the Kron-reduced Laplacian L_r (right panels), for a sequence of graphs with edge probability given by p and increasing number of negative edges (with $P[\text{negative edge}] = p \cdot p_{ne}$, $p_{ne} \in \{0.05, 0.1, \ldots, 0.6\}$). (a): $p = 0.2$. (b): $p = 0.5$.

Proof in Appendix E.

In addition, combining the results of Theorem 4.9 and [10] we have the following 2 sufficient conditions for L to be psd of corank 1:

K1: $-L$ is EEP;

K2: L_r is psd, where $L_r \in \mathbb{R}^{\text{card}(\alpha) \times \text{card}(\alpha)}$ and α is the set of nodes incident to negatively weighted edges.

The following example suggests that the first sufficient condition is significantly less conservative, especially for dense graphs.

Example 6.4. In Figure 2 we consider a sequence of signed connected undirected graphs $\mathcal{G}$ with $n = 100$ nodes, in which the edge weights are drawn from a uniform distribution (where p is the edge probability) and with increasing number of negative edges (proportional to a parameter p_{ne}). In particular, $p = 0.2$ for Fig. 2a and $p = 0.5$ for Fig. 2b, and $P[\text{negative edge}] = p \cdot p_{ne}$, where $p_{ne} \in \{0.05, 0.1, \ldots, 0.6\}$. For each value of p_{ne}, we consider 1000 graphs $\mathcal{G}$, and we compare the conditions K1 and K2. Both conditions are equivalent to L psd; however, as shown in the left panels of Fig. 2, the condition K2 is significantly more conservative than K1, especially for dense graphs (Fig. 2b, left panel). In short, it is not always convenient to determine if L is psd by applying the Kron reduction on the graph and using the Kron-reduced Laplacian L_r (whose size card(α) in shown in the right panels of Fig. 2).

6.3. Hitting and commuting times. Another application of the Laplacian pseudoinverse is in the computation of hitting and commuting times in random walks [31, 20, 6]. The hitting time between two nodes i and j, denoted H_{ij}, corresponds to the average number of node transitions required to reach node j for the first time starting from node i. The commuting time between two nodes i and j, denoted F_{ij}, corresponds to the average number of steps taken in a random walk starting from node i, visiting node j for the first time, and returning back to node i.

In [6] the authors express the hitting and commuting times for (unsigned) digraphs in terms of the pseudoinverse of the normalized Laplacian of the network, the latter defined as $\mathcal{L} := I - \Sigma^{-1} A$. In particular, for a weight balanced (unsigned) digraph, the expected hitting time between node i and j, $i, j \in \{1, \ldots, n\}$, is given by

$$H_{ij} = n \cdot (\mathcal{L}^{\dagger}_{ii} - \mathcal{L}^{\dagger}_{ji}), \tag{6.5}$$

where $\mathcal{L}^\dagger$ is the pseudoinverse of $\mathcal{L}$, while the expected commuting time between nodes i and j, $i, j \in \{1, \dots, n\}$, is given by

$$F_{ij} = H_{ij} + H_{ji} = n \cdot (\mathcal{L}^\dagger_{ii} + \mathcal{L}^\dagger_{jj} - \mathcal{L}^\dagger_{ji} - \mathcal{L}^\dagger_{ij}). \tag{6.6}$$

Comparing with (5.1), it is evident that commuting times are strictly related to effective resistance:

$$F((\mathcal{L}^\dagger)_s) = nR((\mathcal{L}^\dagger)_s) = n\big(D_{(\mathcal{L}^\dagger)_s} \mathbb{1}\mathbb{1}^T + \mathbb{1}\mathbb{1}^T D_{(\mathcal{L}^\dagger)_s} - 2(\mathcal{L}^\dagger)_s\big)$$

and, from (5.4),

$$F(\mathcal{L}^\dagger) = nR(\mathcal{L}^\dagger) = F((\mathcal{L}^\dagger)_s) = n\big(D_{\mathcal{L}^\dagger} \mathbb{1}\mathbb{1}^T + \mathbb{1}\mathbb{1}^T D_{\mathcal{L}^\dagger} - (\mathcal{L}^\dagger + (\mathcal{L}^\dagger)^T)\big).$$

Coherently with (5.1), commuting times can be defined also in terms of $(\mathcal{L}_s)^\dagger$.

It is evident from (6.5) that also hitting times are related to R, as they are essentially "half" of the effective resistance. However, due to the directedness nature of H_{ij}, the only meaningful way to express hitting times is in terms of $\mathcal{L}^\dagger$, and in matrix form it reads

$$H(\mathcal{L}^\dagger) = n(D_{\mathcal{L}^\dagger} \mathbb{1}\mathbb{1}^T - (\mathcal{L}^\dagger)^T).$$

Defining hitting times in terms of $(\mathcal{L}^\dagger)_s$ or $(\mathcal{L}_s)^\dagger$ would lead to meaningless quantities, in which the directionality of the edges is lost, as Example 5.7 shows.

Extending this direction-preserving definition of hitting times (6.5) to signed graphs is however problematic, as $H(\mathcal{L}^\dagger)$ may have negative entries, even when $\mathcal{L}$ is normal. Signed graphs are not suitable objects to describe random walk in Markov chains, as transition probabilities must necessarily be nonnegative. Nevertheless, as long as we deal with unsigned digraphs, all our considerations about hitting times make sense, as Example 5.7 shows.

Example 5.7 (cont'd). Consider again the cycle graph of Fig. 1b with unit edge weights. Observe that in this case $\mathcal{L} = L$ (and hence $\mathcal{L}^\dagger = L^\dagger$, etc.) Computing hitting times according to $(L^\dagger)_s, (L_s)^\dagger$, it is:

$$H_{ij}((L^\dagger)_s) = \begin{cases} \frac{n}{2} & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases}, \qquad H_{ij}((L_s)^\dagger) = (n - |j-i|) \cdot |j-i|$$

i.e., the directionality of the walks along the graph is lost. Instead, computing hitting times according to $L^\dagger$ it is

$$H_{ij}(L^\dagger) = \begin{cases} j-i & \text{if } j \geq i \\ n + (j-i) & \text{if } j < i \end{cases}$$

i.e., $H_{ij}(L^\dagger)$ indeed captures the walk length $i \to j$ along the cycle. These results are illustrated in Fig. 1c for the cycle digraph with $n = 20$ nodes of Fig. 1b.

7. Conclusion. For signed graphs, it is shown in this paper that when the associated Laplacians are EEP and normal, then Laplacians and Laplacian pseudoinverses share the same properties (Perron-Frobenius, marginal stability, and psd of the symmetric part). This class of Laplacians include symmetric (EEP) matrices as a subclass, and in it all objects that can be built on the Laplacian pseudoinverse (effective resistance, equivalent conductance, Kron reduction) are univocally defined. When instead we look at digraphs, then multiple constructions are possible for these objects. Each definition seems to have pros and cons, even though several aspects and applications still require a more thorough analysis.

Appendix A. Proof of Lemma 4.2. Assume that L is weight balanced and of corank 1. Eqs. (4.1a)-(4.1d) are all well-known for L symmetric, and follow easily also for range symmetric matrices. They are proven here only for sake of completeness. Eq. (4.1a) is a consequence of L commuting with $L^\dagger$. As for eq. (4.1b), from $(L^\dagger L)^T = L^\dagger L$ and $\mathcal{N}(L^T) = \mathbb{1}$ (L is weight balanced and of corank 1) it follows that $\mathbb{1}^T L^\dagger = \mathbb{1}^T L^\dagger L L^\dagger = \mathbb{1}^T (L^\dagger L)^T L^\dagger = \mathbb{1}^T L^T (L^\dagger)^T L^\dagger = 0$, i.e., $L^\dagger$ has $\mathbb{1}$ as left eigenvector relative to 0. The proof for the right eigenvector is identical. Concerning eq. (4.1c), from $L^\dagger \mathbb{1} = 0$ it is $L^\dagger \Pi = L^\dagger (I - \frac{\mathbb{1}\mathbb{1}^T}{n}) = L^\dagger$, and similarly for $\Pi L^\dagger = L^\dagger$. For eq. (4.1d), since $L+\gamma J$ is nonsingular, as in [13], it is enough to show the following:

$$(L+\gamma J)(L^\dagger + \frac{1}{\gamma} J) = LL^\dagger + \gamma J L^\dagger + \frac{1}{\gamma} LJ + J^2 = \Pi + J = I - J + J = I,$$

where we have used the properties of Lemma 4.1. Then, $\mathcal{N}(L) = \mathcal{N}(L^T) = \mathcal{N}(L^\dagger) = \mathcal{N}((L^\dagger)^T) = \mathrm{span}(\mathbb{1})$ and (4.1d) imply that $L^\dagger$ is weight balanced of corank 1. Notice that irreducibility of L and $L^\dagger$ follows from Lemma 3.3.

Finally, we need to show that if L is normal then $L^\dagger$ is normal. L normal, J symmetric and $LJ = L^T J = JL = JL^T = 0$ imply $L + \gamma J$ normal, which means that $(L+\gamma J)^{-1}$ is also normal. Since J is symmetric (hence normal) and satisfies the properties of Lemma 4.1, to show that $L^\dagger$ is normal it is sufficient to observe that $(L+\gamma J)^{-1} J = \frac{1}{\gamma} J = J(L+\gamma J)^{-1}$.

Appendix B. Unsigned graph case.

Proof of Theorem 4.4. In Theorem 3.1 it is shown that when $\mathcal{G}(A)$ is unsigned and L is weight balanced, then $L_s = \frac{L+L^T}{2}$ is psd of corank 1. In the following proof, we first show (iii). Then, we prove (iii)$\Longrightarrow$(ii), (ii)$\Longrightarrow$(i), and (ii)$\Longrightarrow$(iv).

(iii) Using equation (4.1d) of Lemma 4.2 we can explicitly write $(L^\dagger)_s$ as follows:

$$\begin{aligned}
(L^\dagger)_s = & \frac{(L+\gamma J)^{-1} + (L^T + \gamma J)^{-1}}{2} - \frac{1}{\gamma} J \\
= & (L+\gamma J)^{-1} \frac{L^T + \gamma J + L + \gamma J}{2} (L^T + \gamma J)^{-1} - \frac{1}{\gamma} J \\
= & (L+\gamma J)^{-1} \left((L_s + \gamma J) - \frac{(L+\gamma J) J (L^T + \gamma J)}{\gamma} \right) (L^T + \gamma J)^{-1} \\
\overset{*}{=} & (L+\gamma J)^{-1} \big((L_s + \gamma J) - \gamma J \big) (L^T + \gamma J)^{-1} \\
= & (L+\gamma J)^{-1} L_s (L+\gamma J)^{-T}.
\end{aligned}$$

In the step marked $*$ we have used the properties of J listed in Lemma 4.1. Hence, since L_s is psd of corank 1 so must be $(L^\dagger)_s$, and, since $\mathcal{N}(L_s) = \mathrm{span}(\mathbb{1})$, then $\mathcal{N}((L^\dagger)_s) = \mathrm{span}(\mathbb{1})$.

(iii)$\Longrightarrow$(ii) If $(L^\dagger)_s$ is psd then all the eigenvalues of $L^\dagger$ must have nonnegative real part, and $L^\dagger$ must be a range symmetric matrix, i.e., $\mathcal{N}(L^\dagger) = \mathcal{N}((L^\dagger)^T)$. Assume by contradiction that $\exists\, v \in \mathcal{N}(L^\dagger), v \notin \mathrm{span}(\mathbb{1})$. Then $(L^\dagger)_s v = 0$, which implies a contradiction since $\mathcal{N}((L^\dagger)_s) = \mathrm{span}(\mathbb{1})$. Hence, $L^\dagger$ must be of corank 1.

(ii)$\Longrightarrow$(i) This statement follows from Theorem 3.6; we report here the proof for completeness. Let $-L^\dagger$ be marginally stable (and weight balanced) of corank 1, i.e.,

$0 = \lambda_1(L^\dagger) < \mathrm{Re}[\lambda_2(L^\dagger)] \leq \cdots \leq \mathrm{Re}[\lambda_n(L^\dagger)]$ and $\mathcal{N}(L^\dagger) = \mathcal{N}((L^\dagger)^T) = \mathrm{span}(\mathbb{1})$. Choosing $d > \max_{i=2,\dots,n} \dfrac{\left|\lambda_i(L^\dagger)\right|^2}{2\mathrm{Re}[\lambda_i(L^\dagger)]}$, $B = dI - L^\dagger$ has $\rho(B) = d$ as a simple eigenvalue of eigenvector $\mathbb{1}$ and so does B^T. Hence $B, B^T \in \mathcal{PF}$, or, from Theorem 2.3, $B \overset{\vee}{>} 0$, i.e., B is eventually positive. Therefore, from Lemma 2.5 $-L^\dagger$ is EEP.

(ii)$\Longrightarrow$(iv) Finally, (iv) holds, i.e., $(L_s)^\dagger$ is psd of corank 1, because $(L_s)^\dagger$ is the pseudoinverse of an unsigned, symmetric, and irreducible Laplacian matrix. ∎

Appendix C. Signed undirected graphs case.

Proof of Theorem 4.8.

(i)$\Longrightarrow$(ii) To show that $L^\dagger$ is psd of corank 1, denote by $\lambda_i(L)$ the eigenvalues of L, of eigenvectors $\mathbb{1}, v_2, \dots v_n$. Using Theorem 3.4, since $-L$ is EEP then L is psd of corank 1, meaning that $0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_n(L)$. Consider eq. (4.1d) of Lemma 4.2. Choosing $\gamma \neq 0$, since J is the orthogonal projection onto $\mathcal{N}(L) = \mathcal{N}(L^T) = \mathrm{span}(\mathbb{1})$, the effect of adding γJ to L is only to shift the 0 eigenvalue to γ, while $\lambda_2(L), \dots, \lambda_n(L)$ are unchanged (see [22, Thm 2.4.10.1]). For the nonsingular $L + \gamma J$ the inverse $(L + \gamma J)^{-1}$ has eigenvalues $\frac{1}{\gamma}, \frac{1}{\lambda_2(L)}, \dots, \frac{1}{\lambda_n(L)}$ of eigenvectors $\mathbb{1}, v_2, \dots v_n$. From orthogonality, $(L + \gamma J)^{-1} - \frac{1}{\gamma} J$ only shifts the $\frac{1}{\gamma}$ eigenvalue back to the origin without touching the other eigenvalues.

(i)$\Longrightarrow$(iii) Assume that $-L$ is EEP, that is, L is psd of corank 1 (see Theorem 3.4). Then $L^\dagger$ is also psd of corank 1, see Lemma 4.2 and proof (i)$\Longrightarrow$(ii). To prove that $-L^\dagger$ is EEP, we can use Theorem 3.4. The proof is here reported for completeness. In particular, from Lemma 4.2, we know that $L^\dagger$ is psd with $0 = \lambda_1(L^\dagger) < \lambda_2(L^\dagger) \leq \cdots \leq \lambda_n(L^\dagger)$ and with $\mathbb{1}$ as left/right eigenvector for 0. If we choose $d > \max_{i=2,\dots,n} \dfrac{\lambda_i(L^\dagger)}{2}$, then $B = dI - L^\dagger$ has $\rho(B) = d$ as a simple eigenvalue of eigenvector $\mathbb{1}$ and so does B^T. Hence $B, B^T \in \mathcal{PF}$, or, from Theorem 2.3, $B \overset{\vee}{>} 0$, i.e., B is eventually positive. Hence from Lemma 2.5 $-L^\dagger$ is EEP.

(iii)$\Longrightarrow$(i) Since $L^\dagger$ is weight balanced of corank 1 with $\mathrm{span}(\mathbb{1}) = \mathcal{N}(L^\dagger) = \mathcal{N}((L^\dagger)^T)$, it is itself a signed Laplacian. The argument can be proven in a similar way as the opposite direction, observing that $L = (L^\dagger)^\dagger$. ∎

Proof of Theorem 4.9. Let α (with $\mathrm{card}(\alpha) \in [2, n-1]$) and $\beta = \{1, \dots, n\} \setminus \alpha$ be a partition of the node set $\mathcal{V}$ meaning that, after an adequate permutation, L can be rewritten as $L = \begin{bmatrix} L[\alpha] & L[\alpha,\beta] \\ L[\beta,\alpha] & L[\beta] \end{bmatrix}$. Let $L_r = L/L[\beta] = L[\alpha] - L[\alpha,\beta]L[\beta]^{-1}L[\beta,\alpha] \in \mathbb{R}^{\mathrm{card}(\alpha)\times\mathrm{card}(\alpha)}$ be the Kron reduced matrix. Note that L_r is symmetric and $\mathbb{1}_{\mathrm{card}(\alpha)} \in \mathcal{N}(L_r)$ (see also [13, Lemma II.1]), meaning that L_r is itself a signed Laplacian.

(i)$\Longrightarrow$(ii)$\Longleftrightarrow$(iii). Assume that $-L$ is EEP or, equivalently, that L is psd of corank 1 (see Theorem 3.4). Then $L[\beta]$ is also psd as it is a principal submatrix of L. In what follows we prove first, by contradiction, that L irreducible and psd of corank 1 imply that $L[\beta]$ is actually pd. Then, we show that L_r is psd of corank 1.

Let $\mathrm{card}(\beta) = 1$ and assume, by contradiction, that $L[\beta] = 0$. However, L psd means that L has the row and column inclusion property, i.e., if the diagonal element $L[\beta]$ is zero then $L[\alpha,\beta] = 0$ and $L[\beta,\alpha] = 0$, which contradicts the hypothesis that L is irreducible. Hence, $L[\beta] > 0$ (pd). Now we repeat the same argument for $1 < \mathrm{card}(\beta) \leq n-2$: suppose by contradiction that $\exists\, v \in \mathbb{R}^{\mathrm{card}(\beta)}$ s.t. $L[\beta]v = 0$ (i.e., $L[\beta]$ is not pd). Then $\bar{v} = \begin{bmatrix} 0 \\ v \end{bmatrix}$ is s.t. $L\bar{v} = 0$ (since $\bar{v}^T L \bar{v} = 0$), which contradicts the hypothesis that L has corank 1 since $\mathbb{1} \in \mathcal{N}(L)$ and $\bar{v} \notin \text{span}(\mathbb{1})$ (notice that if $v = \mathbb{1}_{\text{card}(\beta)}$, then either $L[\beta,\alpha]$ is the zero matrix - in contradiction with the hypothesis that L is irreducible -, or $\begin{bmatrix} \mathbb{1}_{\text{card}(\alpha)} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbb{1}_{\text{card}(\beta)} \end{bmatrix} \in \mathcal{N}(L)$ - in contradiction with the hypothesis that L has corank 1). Therefore, $L[\beta]$ is pd.

Rewrite L as follows, where $L[\alpha,\beta]L[\beta]^{-1} = (L[\beta]^{-1}L[\beta,\alpha])^T$:

$$L = \begin{bmatrix} I & L[\alpha,\beta]L[\beta]^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} L_r & 0 \\ 0 & L[\beta] \end{bmatrix} \begin{bmatrix} I & 0 \\ L[\beta]^{-1}L[\beta,\alpha] & I \end{bmatrix},$$

Applying Sylverster's law of inertia, L psd of corank 1 and $L[\beta]$ pd imply L_r psd of corank 1 or, equivalently (from Theorem 3.4), $-L_r$ EEP.

(i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii). Let α be the set of nodes incident to negatively weighted edges. In what follows, the steps marked by the symbol $\star$ follow from Theorem 3.4 while the step marked by the symbol $\triangle$ from [10, Theorem 1]:

$$-L \text{ EEP} \overset{\star}{\Longleftrightarrow} L \text{ psd of corank 1} \overset{\triangle}{\Longleftrightarrow} L_r \text{ psd of corank 1} \overset{\star}{\Longleftrightarrow} -L_r \text{ EEP}. \qquad \square$$

Proof of Lemma 4.11. From $L\mathbb{1} = (k_1L_1 + k_2L_2)\mathbb{1} = k_1L_1\mathbb{1} + k_2L_2\mathbb{1} = 0$, it follows that L is a signed Laplacian. Since L_1, L_2 are psd and $k_1, k_2 > 0$, then

$$x^TLx = x^T(k_1L_1 + k_2L_2)x = k_1x^TL_1x + k_2x^TL_2x \geq 0$$

that is, L is psd, and

$$x^TLx = 0 \Longleftrightarrow \begin{cases} x^TL_1x = 0 \\ x^TL_2x = 0 \end{cases} \Longleftrightarrow x = \text{span}(\mathbb{1}),$$

that is, L is of corank 1. Applying Theorem 3.4, L psd of corank 1 implies $-L$ EEP which concludes the proof. $\square$

Appendix D. Signed directed graphs case.

Proof of Theorem 4.12.

(i)$\Longleftrightarrow$(iii) The proof follows the proof of Theorem 4.8, with the difference that marginal stability of the Laplacian and its pseudoinverse has to be considered instead of positive semidefiniteness. An important observation, implied by eq. (4.1d) of Lemma 4.2, is that the eigenvalues of L and $L^\dagger$ are such that

$$\lambda_1(L^\dagger) = \lambda_1(L) = 0$$

and, for each $i = 2, \ldots, n$, there exists a (unique) $k = 2, \ldots, n$ (and viceversa) s.t.

$$\lambda_i(L^\dagger) = \frac{1}{\lambda_k(L)} \Longrightarrow \text{Re}[\lambda_i(L^\dagger)] = \frac{\text{Re}[\lambda_k(L)]}{|\lambda_k(L)|^2}.$$

Note that the reason behind different subscripts i and k is that we are assuming that the eigenvalues of L and $L^\dagger$ are ordered in a nondecreasing manner and, for instance, $\text{Re}[\lambda_i(L)] \leq \text{Re}[\lambda_j(L)] \not\Longrightarrow \text{Re}[\lambda_i(L^\dagger)] \leq \text{Re}[\lambda_j(L^\dagger)]$. If $-L^\dagger$ is marginally stable with corank 1, then $B = dI - L^\dagger \in \mathcal{PF}$ with $d > \max_{i=2,\ldots,n} \frac{|\lambda_i(L^\dagger)|^2}{2\text{Re}[\lambda_i(L^\dagger)]} = \max_{i=2,\ldots,n} \frac{1}{2\text{Re}[\lambda_i(L)]}$. Therefore, from Lemma 2.5, $-L^\dagger$ is EEP.

(iv) Assume that L is normal or, equivalently, that $L^\dagger$ is normal (see Lemma 4.2). Since L normal implies L weight balanced, the statements (i), (ii), and (iii) are still equivalent. To show the equivalence with (iv), it is sufficient to apply Theorem 3.6 on $L^\dagger$ since $L^\dagger$ is itself a normal signed Laplacian of corank 1.

(v) Similarly to (iv), under the assumption that L is normal, the result follows directly from Theorem 3.6 since $(L_s)^\dagger$ is the pseudoinverse of a symmetric signed Laplacian which is psd of corank 1. ∎

Proof of Remark 4.13. If L is normal (and of corank 1), then there exists an orthonormal matrix U such that $L = UDU^T$, with

$$D = \mu_1 \oplus \cdots \oplus \mu_{n-2\ell} \oplus \begin{bmatrix} \nu_1 & \omega_1 \\ -\omega_1 & \nu_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \nu_\ell & \omega_\ell \\ -\omega_\ell & \nu_\ell \end{bmatrix}$$

where $\mu_1, \dots, \mu_{n-2\ell}$ are the real eigenvalues of L and $\nu_1 \pm i\omega_1, \dots, \nu_\ell \pm i\omega_\ell$ are its complex conjugate eigenvalues (with $\ell \in [0, \lfloor \frac{n}{2} \rfloor])$, and $\oplus$ indicates direct sum. Without lack of generality, assume that the first column of U is $\frac{\mathbb{1}}{\sqrt{n}}$, which means that $\mu_1 = 0$ and $D = 0 \oplus \bar{D}$, where

$$\bar{D} = \mu_2 \oplus \cdots \oplus \mu_{n-2\ell} \oplus \begin{bmatrix} \nu_1 & \omega_1 \\ -\omega_1 & \nu_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \nu_\ell & \omega_\ell \\ -\omega_\ell & \nu_\ell \end{bmatrix} \tag{D.1}$$

is nonsingular. Then, $L_s = U(0 \oplus \frac{\bar{D}+\bar{D}^T}{2})U^T$ and $L^\dagger = U(0 \oplus \bar{D}^{-1})U^T$, yielding

$$(L^\dagger)_s = U \begin{bmatrix} 0 & 0 \\ 0 & \frac{\bar{D}^{-1}+\bar{D}^{-T}}{2} \end{bmatrix} U^T \neq U \begin{bmatrix} 0 & 0 \\ 0 & (\frac{\bar{D}+\bar{D}^T}{2})^{-1} \end{bmatrix} U^T = (L_s)^\dagger.$$

∎

Proof of Lemma 4.14. Assume that L is normal and $-L$ is EEP, i.e., $-L$ is marginally stable of corank 1. In the first part of the proof we write an explicit expression for $(L_s)^\dagger$, while in the second part of the proof we show that the matrix X of eq. (4.3) is equal to $(L_s)^\dagger$.

Using the same notation introduced in the proof of Remark 4.13, since $L\mathbb{1} = L^T\mathbb{1} = 0$ and L normal, then there exists an orthonormal matrix U such that $L = U(0 \oplus \bar{D})U^T$ where $\bar{D}$ is given by (D.1). In particular, U can be chosen as $U = \left[\frac{\mathbb{1}}{\sqrt{n}} \;\; Q^T\right]$, where $Q \in \mathbb{R}^{n-1\times n}$ satisfies

$$Q\mathbb{1}_n = 0, \quad QQ^T = I_{n-1}, \quad Q^TQ = I - \frac{\mathbb{1}\mathbb{1}^T}{n} = \Pi. \tag{D.2}$$

Let $\Lambda := \frac{\bar{D}+\bar{D}^T}{2} = \operatorname{diag}(\mu_2, \dots, \mu_{n-2\ell}, \nu_1, \nu_1, \dots, \nu_\ell, \nu_\ell)$. Then, since $L = Q^T\bar{D}Q$, the pseudoinverse of its symmetric part is given by

$$(L_s)^\dagger = \left(Q^T \frac{\bar{D}+\bar{D}^T}{2} Q\right)^\dagger = (Q^T\Lambda Q)^\dagger = Q^T\Lambda^{-1}Q.$$

To calculate X, defined in eq. (4.3), we need to define first a reduced Laplacian matrix $\bar{L}$, and then find the solution S of the Lyapunov equation $\bar{L}S + S\bar{L}^T = I_{n-1}$. Here we use the fact that, even if $\bar{L}$ is not unique (since it depends on the choice of Q), the computation of X in eq. (4.3) is independent of the choice of Q [41]. Therefore, we choose the matrix Q introduced previously in the definition of $(L_s)^\dagger$ and, by construction, we obtain that

$$\bar{L} = QLQ^T = Q(Q^T\bar{D}Q)Q^T = \bar{D}$$

is a projection of L onto $\mathbb{1}^\perp$, and that $-\bar{L}$ is Hurwitz. Then, $S = \frac{1}{2}\Lambda^{-1}$, is the unique solution of the Lyapunov equation $-\bar{L}S + S(-\bar{L}^T) = -I_{n-1}$. Therefore,

$$X = 2Q^T SQ = Q^T\Lambda^{-1}Q = (L_s)^\dagger. \qquad \square$$

Proof of Lemma 4.16. Assume that L is weight balanced and $-L$ is EEP, and consider the matrix $L_\alpha := \alpha L + (1-\alpha)\Pi$, $0 \le \alpha \le 1$. The matrix Π is a symmetric, unsigned Laplacian matrix, and $-\Pi$ is EEP/marginally stable of corank 1. Since $\mathcal{N}(L) = \mathcal{N}(L^T) = \mathcal{N}(\Pi) = \text{span}(\mathbb{1})$, then L_α is also a signed, weight balanced Laplacian such that $\mathcal{N}(L_\alpha) = \mathcal{N}(L_\alpha^T) = \text{span}(\mathbb{1})$, $\lambda_1(L_\alpha) = 0$, and $\lambda_i(L_\alpha) = \alpha\lambda_i(L) + (1-\alpha)$ for all $i = 2, \dots, n$. Hence, $\text{Re}[\lambda_i(L_\alpha)] > 0$ for all i and $\alpha \in [0,1]$, which means that $-L_\alpha$ is marginally stable of corank 1 and, therefore (see Theorem 3.6), EEP. $\square$

Appendix E. Applications.

Proof of Lemma 5.2. Theorem 4.12 shows that for a signed digraph with normal Laplacian L s.t. $-L$ is EEP, the matrices L_s, $(L_s)^\dagger$ and $(L^\dagger)_s$ are themselves signed Laplacians and they are psd of corank 1 with $\mathcal{N}(L_s) = \mathcal{N}((L_s)^\dagger) = \mathcal{N}((L^\dagger)_s) = \text{span}(\mathbb{1})$. Since $R_{ij}(X)$ is a quadratic form generated by $X \in \{(L_s)^\dagger, (L^\dagger)_s\}$, then

$$\begin{aligned} R_{ij}(X) &= (e_i - e_j)^T X(e_i - e_j) = \|X^{\frac{1}{2}}(e_i - e_j)\|_2^2 \\ &= \|X^{\frac{1}{2}}(e_j - e_i)\|_2^2 = (e_j - e_i)^T X(e_j - e_i) = R_{ji}(X) \\ \text{and } R_{ij}(X) &= (e_i - e_j)^T X(e_i - e_j) = \|X^{\frac{1}{2}}(e_i - e_j)\|_2^2 \ge 0 \end{aligned}$$

for all $i, j = 1, \dots, n$, with $R_{ij}(X) = 0$ if and only if $i = j$ (since $e_i - e_j \in \text{span}(\mathbb{1}^\perp)$ when $i \neq j$). The triangle inequality holds since, for all $i, j, k = 1, \dots, n$, it is:

$$\begin{aligned} \sqrt{R_{ik}(X)} + \sqrt{R_{kj}(X)} &= \|X^{\frac{1}{2}}(e_i - e_k)\|_2 + \|X^{\frac{1}{2}}(e_k - e_j)\|_2 \\ &\ge \|X^{\frac{1}{2}}(e_i - e_k) + X^{\frac{1}{2}}(e_k - e_j)\|_2 = \|X^{\frac{1}{2}}(e_i - e_j)\|_2 = \sqrt{R_{ij}(X)} \end{aligned}$$

Finally, to prove that R is a Euclidean distance matrix we need to show that $z^T R(X)z \le 0 \; \forall\, z \perp \mathbb{1}$. Since $X \in \{(L_s)^\dagger, (L^\dagger)_s\}$ is psd with $\mathcal{N}(X) = \text{span}(\mathbb{1})$, then:

$$z^T R(X)z = z^T(D_X\mathbb{1}\mathbb{1}^T + \mathbb{1}\mathbb{1}^T D_X - 2X)z = -2z^T Xz \le 0 \quad \forall\, z \perp \mathbb{1}. \qquad \square$$

Proof of Lemma 5.4.

(i) We use the notation introduced in the proofs of Remark 4.13 and Lemma 4.14 to rewrite $(L^\dagger)_s$ and $(L_s)^\dagger$:

$$(L^\dagger)_s = Q^T\Big(\frac{\bar{D}^{-1} + \bar{D}^{-T}}{2}\Big)Q, \quad (L_s)^\dagger = Q^T\Big(\frac{\bar{D} + \bar{D}^T}{2}\Big)^{-1}Q$$

where Q satisfies (D.2) and $\bar{D}$ is given by (D.1), i.e.,

$$\bar{D} = \mu_2 \oplus \cdots \oplus \mu_{n-2\ell} \oplus \begin{bmatrix} \nu_1 & \omega_1 \\ -\omega_1 & \nu_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \nu_\ell & \omega_\ell \\ -\omega_\ell & \nu_\ell \end{bmatrix}$$

with $\mu_2 > 0, \dots, \mu_{n-2\ell} > 0, \nu_1 > 0, \dots, \nu_\ell > 0$. Therefore:

$$\begin{aligned} \frac{\bar{D}^{-1} + \bar{D}^{-T}}{2} &= \text{diag}\left(\frac{1}{\mu_2}, \dots, \frac{1}{\mu_{n-2\ell}}, \frac{\nu_1}{\nu_1^2 + \omega_1^2}, \frac{\nu_1}{\nu_1^2 + \omega_1^2}, \dots, \frac{\nu_\ell}{\nu_\ell^2 + \omega_\ell^2}, \frac{\nu_\ell}{\nu_\ell^2 + \omega_\ell^2}\right) \\ \Big(\frac{\bar{D} + \bar{D}^T}{2}\Big)^{-1} &= \text{diag}\left(\frac{1}{\mu_2}, \dots, \frac{1}{\mu_{n-2\ell}}, \frac{1}{\nu_1}, \frac{1}{\nu_1}, \dots, \frac{1}{\nu_\ell}, \frac{1}{\nu_\ell}\right). \end{aligned}$$

Observe that the diagonal matrix

$$\left(\frac{\bar{D}+\bar{D}^T}{2}\right)^{-1} - \frac{\bar{D}^{-1}+\bar{D}^{-T}}{2} \tag{E.1}$$

has nonnegative diagonal elements (i.e., it is psd) since $\frac{1}{\nu_i} \geq \frac{\nu_i}{\nu_i^2+\omega_i^2}$ for all i.

The difference between the effective resistances calculated according to $(L_s)^\dagger$ and $(L^\dagger)_s$ is given by:

$$\begin{aligned} R_{ij}((L_s)^\dagger) - R_{ij}((L^\dagger)_s) &= (e_i-e_j)^T (L_s)^\dagger (e_i-e_j) - (e_i-e_j)^T (L^\dagger)_s (e_i-e_j) \\ &= (e_i-e_j)^T \big((L_s)^\dagger - (L^\dagger)_s\big)(e_i-e_j) \\ &= (e_i-e_j)^T Q^T \left(\left(\frac{\bar{D}+\bar{D}^T}{2}\right)^{-1} - \frac{\bar{D}^{-1}+\bar{D}^{-T}}{2}\right) Q(e_i-e_j) \geq 0 \end{aligned}$$

since the matrix in eq. (E.1) is psd. Therefore, $R_{ij}((L_s)^\dagger) \geq R_{ij}((L^\dagger)_s)$ for all i, j.

(ii) From Theorem 4.12, L normal and $-L$ EEP mean that both $(L_s)^\dagger$ and $(L^\dagger)_s$ are psd of corank 1, and $\mathcal{N}((L_s)^\dagger) = \mathcal{N}((L^\dagger)_s) = \operatorname{span}(\mathbb{1})$. Hence, for $X \in \{(L_s)^\dagger, (L^\dagger)_s\}$, it holds that $\mathbb{R}(X)\mathbb{1} = nD_X\mathbb{1} + (\mathbb{1}^T D_X \mathbb{1})\mathbb{1}$, which implies $R_{\text{tot}}(X) = \frac{1}{2}\mathbb{1}^T\mathbb{R}(X)\mathbb{1} = n\cdot(\mathbb{1}^T D_X\mathbb{1}) = n \cdot \operatorname{Tr}(X)$, since D_X contains the diagonal elements of X. The matrix $(L_s)^\dagger$ is symmetric, which means that $\lambda_i((L_s)^\dagger) = \frac{1}{\lambda_i(L_s)}$ and, since L is normal, $\lambda_i(L_s) = \operatorname{Re}[\lambda_i(L)]$ for all $i = 2, \ldots, n$. Therefore,

$$R_{\text{tot}}((L_s)^\dagger) = n \cdot \operatorname{Tr}\big((L_s)^\dagger\big) = n\sum_{i=2}^n \lambda_i((L_s)^\dagger) = n\sum_{i=2}^n \frac{1}{\lambda_i(L_s)} = n\sum_{i=2}^n \frac{1}{\operatorname{Re}[\lambda_i(L)]}.$$

Similarly, since $L^\dagger$ is normal, $\lambda_i((L^\dagger)_s) = \operatorname{Re}[\lambda_i(L^\dagger)]$ for all $i = 2, \ldots, n$. Therefore,

$$R_{\text{tot}}((L^\dagger)_s) = n \cdot \operatorname{Tr}\big((L^\dagger)_s\big) = n\sum_{i=2}^n \operatorname{Re}[\lambda_i(L^\dagger)].$$

Finally, since $\lambda_i(L^\dagger) = \frac{1}{\lambda_i(L)}$, we obtain:

$$R_{\text{tot}}((L^\dagger)_s) = n\sum_{i=2}^n \operatorname{Re}\Big[\frac{1}{\lambda_i(L)}\Big] = n\sum_{i=2}^n \frac{\operatorname{Re}[\lambda_i(L)]}{|\lambda_i(L)|^2} \leq n\sum_{i=2}^n \frac{1}{\operatorname{Re}[\lambda_i(L)]} = R_{\text{tot}}((L_s)^\dagger).$$

□

Proof of Lemma 5.8. Let $\Sigma_i = \operatorname{diag}(A_i\mathbb{1})$, $L_i = \Sigma_i - A_i$, $i = 1, 2$. If $A_1 \geq A_2$ then $\Sigma_1 \geq \Sigma_2$. It also holds that $A_{1s} = \frac{A_1 + A_1^T}{2} \geq A_{2s} = \frac{A_2+A_2^T}{2}$ or, equivalently, that $A_s := A_{1s} - A_{2s} \geq 0$. Notice that $\Sigma := \Sigma_1 - \Sigma_2 = \operatorname{diag}(A_s\mathbb{1})$ is a diagonal matrix with nonnegative elements on the diagonal. Define $L_s := \Sigma - A_s$, which is the (symmetric) Laplacian corresponding to the undirected nonnegative graph $\mathcal{G}(A_s)$: L_s may be reducible but it is psd since $A_s \geq 0$. Hence $0 = \lambda_1(L_s) \leq \lambda_j(L_s)$ for all j.

Rewriting $L_{1s} := \frac{L_1+L_1^T}{2}$ in terms of $L_{2s} := \frac{L_2+L_2^T}{2}$ and L_s, i.e., $L_{1s} = L_{2s} + L_s$, we can apply the monotonicity theorem [22, Corollary 4.3.12] and state that $\lambda_k(L_{1s}) = \lambda_k(L_{2s} + L_s) \geq \lambda_k(L_{2s})$ for all $k = 2, \ldots, n$. Therefore, it follows that:

$$\begin{aligned} R_{\text{tot}}((L_{1s})^\dagger) &= n \cdot \operatorname{Tr}\big((L_{1s})^\dagger\big) = n\sum_{i=2}^n \lambda_i((L_{1s})^\dagger) = n\sum_{i=2}^n \frac{1}{\lambda_i(L_{1s})} \\ &\leq n\sum_{i=2}^n \frac{1}{\lambda_i(L_{2s})} = n\sum_{i=2}^n \lambda_i((L_{2s})^\dagger) = n\cdot\operatorname{Tr}\big((L_{2s})^\dagger\big) = R_{\text{tot}}((L_{2s})^\dagger). \end{aligned}$$

□

Proof of Lemma 6.3. After an adequate permutation, the Laplacian L of the graph $\mathcal{G}(A)$ can be rewritten as $L = \begin{bmatrix} L[\alpha] & L[\alpha,\beta] \\ L[\beta,\alpha] & L[\beta] \end{bmatrix}$, and it holds that

$$\begin{bmatrix} L_r & \\ & L[\beta] \end{bmatrix} = \begin{bmatrix} I & -L[\alpha,\beta]L[\beta]^{-1} \\ 0 & I \end{bmatrix} L \begin{bmatrix} I & 0 \\ -L[\beta]^{-1}L[\beta,\alpha] & I \end{bmatrix}.$$

To compute $(L_r)^\dagger$ we use the identities [36] $(XYZ)^\dagger = (X^\dagger XYZ)^\dagger Y (XYZZ^\dagger)^\dagger$ and $(XY)^\dagger = (X^\dagger XY)^\dagger (XYY^\dagger)^\dagger$, obtaining:

$$\begin{aligned} \begin{bmatrix} (L_r)^\dagger & \\ & L[\beta]^{-1} \end{bmatrix} &= \left(L \begin{bmatrix} I & 0 \\ -L[\beta]^{-1}L[\beta,\alpha] & I \end{bmatrix} \right)^\dagger L \left(\begin{bmatrix} I & -L[\alpha,\beta]L[\beta]^{-1} \\ 0 & I \end{bmatrix} L \right)^\dagger \\ &= \left(\Pi \begin{bmatrix} I & 0 \\ -L[\beta]^{-1}L[\beta,\alpha] & I \end{bmatrix} \right)^\dagger L^\dagger \left(\begin{bmatrix} I & -L[\alpha,\beta]L[\beta]^{-1} \\ 0 & I \end{bmatrix} \Pi \right)^\dagger \\ &= \begin{bmatrix} \Pi[\alpha] & 0 \\ L[\beta]^{-1}L[\beta,\alpha] & I \end{bmatrix} L^\dagger \begin{bmatrix} \Pi[\alpha] & L[\alpha,\beta]L[\beta]^{-1} \\ 0 & I \end{bmatrix} \end{aligned}$$

that is, $(L_r)^\dagger = \Pi[\alpha]\, L^\dagger[\alpha,\alpha]\, \Pi[\alpha]$. Then, given two nodes $i, j \in \alpha$, it holds that:

$$\begin{aligned} R_{ij}((L_r)^\dagger) &= (e_i[\alpha] - e_j[\alpha])^T (L_r)^\dagger (e_i[\alpha] - e_j[\alpha]) \\ &= (e_i[\alpha] - e_j[\alpha])^T \Pi[\alpha]\, L^\dagger[\alpha,\alpha]\, \Pi[\alpha] (e_i[\alpha] - e_j[\alpha]) \\ &= (e_i[\alpha] - e_j[\alpha])^T\, L^\dagger[\alpha,\alpha]\, (e_i[\alpha] - e_j[\alpha]) = (e_i - e_j)^T L^\dagger\, (e_i - e_j) = R_{ij}(L^\dagger) \end{aligned}$$ ∎

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