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Blazej M. Szablikowski, Maciej Blaszak and Krzysztof Marciniak

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Błażej M. Szablikowski*†, Maciej Błaszak†, and Krzysztof Marciniak‡

†Faculty of Physics, Adam Mickiewicz University
Uniwersytetu Poznańskiego 2, 61-614 Poznań, Poland
bszablik@amu.edu.pl, blaszakm@amu.edu.pl

‡Department of Science and Technology
Linköping University, Campus Norrköping
601 74 Norrköping, Sweden
krzma@itn.liu.se

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Abstract

In this article we investigate stationary coupled Korteweg–de Vries (cKdV) systems and prove that every N-field stationary cKdV system can be written, after a careful reparametrization of jet variables, as a classical separable Stäckel system in $N+1$ different ways. For each of these $N+1$ parametrizations we present an explicit map between the jet variables and the separation variables of the system. Finally, we show that each pair of Stäckel representations of the same stationary cKdV system, when considered in the phase space extended by Casimir variables, is connected by an appropriate finite-dimensional Miura map, which leads to an $(N+1)$-Hamiltonian formulation for the stationary cKdV system.

Keywords: cKdV hierarchy, DWW hierarchy, stationary flows, Stäckel systems, Miura maps

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*Corresponding author.

1 Introduction

Since the classical works of Bogoyavlensky, Novikov [14] and Mokhov [26] there has been a tremendous amount of research devoted to connections between soliton hierarchies and their integrable finite-dimensional reductions (see for example the bibliography in the survey [5] as well as the introductory part in [13]). In [2, 3] the authors presented a fairly general construction of finite-dimensional completely integrable Hamiltonian systems obtained by reductions of the Korteweg–de Vries (KdV) and the coupled Korteweg–de Vries (cKdV) soliton hierarchies to their stationary and restricted flows.

In this article we thoroughly investigate the concept of stationary cKdV systems and their Stäckel representations, obtaining new results in this area, as described below.

In Section 2 we remind the construction of the N-component cKdV hierarchy $\boldsymbol{u}_{t_n} = \boldsymbol{K}_n[\boldsymbol{u}]$ from the energy-dependent Schrödinger spectral problem [1, 25]. We present these – classical – results in a novel setting of symmetric bilinear differential operators $\mathcal{J}_i$, defined in (2.12), which allows for presenting the pure algebraic recursion formulas for conserved densities (see Proposition 2.1 and formula (2.19)). In Subsection 2.3 we present the integrated form of kernels of all $N + 1$ Hamiltonian operators $\mathbb{B}_m$ of the cKdV hierarchy (see Proposition 2.2) (the N-component cKdV hierarchy is $N + 1$ Hamiltonian) in the language of bi-linear operators $\mathcal{J}_i$. Both these results are new. In Subsection 2.4 we remind the reader the structure of the Lax formulation of cKdV hierarchy.

In Section 3 we study the main object of this paper: the stationary cKdV system. This is a finite-dimensional system originating by restricting the cKdV hierarchy to one of its stationary manifolds $\mathcal{M}_n = \{[\boldsymbol{u}] : K_{n+1}[\boldsymbol{u}] = 0\}$. In Subsection 3.1 we explain how each Hamiltonian operator $\mathbb{B}_m$ generates a Hamiltonian foliation of the $(2n + N)$-dimensional stationary manifold $\mathcal{M}_n$ into $2n$-dimensional leaves. These Hamiltonian foliations are transversal to each other. In Subsection 3.2 we introduce another set of $N + 1$ foliations of $\mathcal{M}_n$ into $2n$-dimensional leaves, each related with imposing a particular stationary constraint on Lax representation of the cKdV hierarchy. We call these foliations Stäckel foliations of $\mathcal{M}_n$. We conclude this subsection by proving that Hamiltonian foliation and Stäckel foliation for the same m coincide (see Theorem 3.2). In Subsection 3.3 we present the Lax representation of the stationary cKdV system on leaves of each of Stäckel foliations and then we show how these foliations lead in a very natural way to separation (spectral) curves of Stäckel systems.

Section 4 contains some basic facts about Stäckel systems and their Lax formulation. This section is introduced in order to keep the article self-contained.

In Section 5, comparing the Lax formulation of the stationary cKdV system as given in Subsection 3.3 with the Lax formulation of Stäckel systems as given in Section 4, we prove the main theorem of this article (Theorem 5.1), stating that each stationary cKdV system can be represented as a Stäckel system on $\mathcal{M}_n$ on $N + 1$ ways.

In Section 6 we prove that all Stäckel representations of the stationary cKdV systems obtained in Section 5, are connected by a corresponding Miura map. This yields immediately $(N + 1)$-Hamiltonian formulations of the cKdV system on $\mathcal{M}_n$. Thus, in this article we demonstrate that the $(N + 1)$-Hamiltonian structure of the cKdV hierarchy generates $(N + 1)$-Hamiltonian structure of the stationary cKdV system on $\mathcal{M}_n$. In consequence, the leaves of a m-th Stäckel foliation become symplectic leaves of the corresponding m-th Hamiltonian operator of the stationary cKdV system.

Section 7 is devoted to examples. The subsection 7.1 focuses on the Dispersive Water Waves (DWW) hierarchy (so that $N = 2$) and its first part contains a very detailed presentation of all main formulas and ingredients for the case $n = 2$. The second part of this subsection presents the case $n = 3$ and $m = 0$. The subsection 7.2 is devoted to the case $N = 4$, $n = 2$ and $m = 0$. This last example is non-generic since $n + m < N - 1$.

In article [13] we performed a similar analysis to described above, but for the KdV case, i.e. the one-component case.

Recently, in [23, 24], the authors revisited the idea of the stationary cKdV system. Their results focused mainly on the $N = 2$ and $N = 3$ case and on first two (i.e. lowest) flows of the hierarchy, while our analysis is general, i.e. valid for all N and all n. Besides, they do not consider the separability problem in the general setting, intensively studied in our article.

Let us also mention that a new approach, partly related to our results, based on the Nijenhuis geometry applicable to multi-component integrable equations, i.e. of KdV type, was proposed recently in [15, 16].

Let us mention that the construction inverse to the one presented in this article is also possible: starting from a carefully chosen family of Stäckel systems one can reconstruct the related hierarchies of stationary systems and hence reconstruct the associated soliton hierarchy. The idea of such a construction appeared for the first time in 1999 during a visit of one of the authors (M.B.) to A.P. Fordy in Leeds University. This idea was then explored for the first time in [8–10].

Finally, let us also mention that a similar idea, linking Stäckel systems with dispersionless field systems was introduced in papers by Ferapontov and Fordy [19–21] and the paper [22].

2 Coupled KdV hierarchy

In this section we review, following [1, 25], and develop the classical construction of the cKdV hierarchy from the energy dependent Schrödinger spectral problem as well its multi-Hamiltonian representation.

2.1 Energy-dependent Schrödinger spectral problem

The N-component cKdV hierarchy originates as the compatibility condition of the energy dependent Schrödinger spectral problem with the appropriate evolutionary part:

$$\begin{aligned} &\psi_{xx} + \mathbb{Q}\psi = 0, \\ &\psi_{t_k} = \frac{1}{2}\mathbb{P}_k\psi_x - \frac{1}{4}(\mathbb{P}_k)_x\psi, \qquad k = 1, 2, \ldots, \end{aligned} \tag{2.1}$$

where

$$\mathbb{Q} := \sum_{i=0}^{N} u_i\lambda^i, \qquad u_N \equiv -1.$$

Here and below, $u_i = u_i(x, t_1, t_2, \ldots)$ are the dynamical fields, while $\mathbb{P}_k$ are so far unspecified functions of the spectral parameter λ and jet variables in u_i. The compatibility conditions $(\psi_{xx})_{t_k} = (\psi_{t_k})_{xx}$ of (2.1) yield the following hierarchy of evolution equations

$$\mathbb{Q}_{t_k} = \frac{1}{4}(\mathbb{P}_k)_{3x} + \mathbb{Q}(\mathbb{P}_k)_x + \frac{1}{2}\mathbb{Q}_x\mathbb{P}_k \equiv J\mathbb{P}_k, \qquad k = 1, 2, \ldots \tag{2.2}$$

where

$$J \equiv \sum_{i=0}^{N} J_i\lambda^i, \qquad J_i := \frac{1}{4}\delta_{i0}\partial_x^3 + u_i\partial_x + \frac{1}{2}(u_i)_x. \tag{2.3}$$

Further, in accordance with [1, 25], we assume that each $\mathbb{P}_k$ is a polynomial of order $k-1$ in λ:

$$\mathbb{P}_k = \sum_{i=0}^{k-1} P_{k-1-i}\lambda^i \equiv P_0\lambda^{k-1} + \ldots + P_{k-2}\lambda + P_{k-1}. \tag{2.4}$$

The conditions for the coefficients P_i in (2.4) can be obtained by requiring consistency of the evolution equations (2.2). It turns out that the coefficients P_i in (2.2) or (2.4) actually do *not* depend on k and that they satisfy the equation

$$J\mathcal{P} \equiv \frac{1}{4}\mathcal{P}_{3x} + \mathbb{Q}\mathcal{P}_x + \frac{1}{2}\mathbb{Q}_x\mathcal{P} = 0, \tag{2.5}$$

where $\mathcal{P}$ is the Laurent series in λ:

$$\mathcal{P} = \sum_{i=0}^{\infty} P_i\lambda^{-i}, \tag{2.6}$$

and thus

$$\mathbb{P}_k = \left[\lambda^{k-1}\mathcal{P}\right]_{\geqslant 0},$$

where $[\cdot]_{\geqslant 0}$ means the projection on the part polynomial in λ.

Notice that

$$J\mathcal{P} = \sum_{i=0}^{N}\sum_{j=0}^{\infty} J_iP_j\lambda^{i-j} \equiv \sum_{k=-\infty}^{N}\ \sum_{i=\max\{0,k\}}^{N} J_iP_{i-k}\lambda^k$$

and thus a straightforward consequence of (2.5) is the equality

$$\sum_{i=0}^{N} J_iP_{i-k} = 0, \qquad \text{where} \quad k \leqslant N. \tag{2.7}$$

To simplify the notation, in (2.7) and further we assume that $P_i = 0$ for $i < 0$.[1] The condition (2.5) not only provides us with (differential) equations on the coefficients P_i in (2.4) or (2.6), but also assures the consistency of the equations from the hierarchy (2.2).

There exists an alternative description of the cKdV hierarchy within the above scheme. If we define

$$\bar{\mathbb{P}}_k := \left[\lambda^{k-1}\mathcal{P}\right]_{<0} \equiv \lambda^{k-1}\mathcal{P} - \mathbb{P}_k = \sum_{i=k}^{\infty} P_i\lambda^{k-1-i}$$

then, due to (2.5), $J\mathbb{P}_k = -J\bar{\mathbb{P}}_k$, and thus the hierarchy (2.2) can alternatively be written as

$$\mathbb{Q}_{t_k} = -J\bar{\mathbb{P}}_k. \tag{2.8}$$

Consequently, the cKdV hierarchy can be defined by (2.2), or equivalently by (2.8), provided that the condition (2.5) holds. The members of the cKdV hierarchy have the form of mutually commuting N-component evolution equations,

$$\boldsymbol{u}_{t_k} = \boldsymbol{K}_k, \qquad k = 1, 2, \ldots, \tag{2.9}$$

defined on an infinite-dimensional functional (smooth) manifold $\mathcal{F}$. Coordinates on $\mathcal{F}$ are given by jet variables $[\boldsymbol{u}] := (\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots)$, with the (field) vector $\boldsymbol{u} := (u_0, \ldots, u_{N-1})^T$. More explicitly, by (2.2) and (2.8), the evolution equations (2.9) can be written in the following two equivalent ways:

$$(u_{i-1})_{t_k} = (\boldsymbol{K}_k)_i \equiv \sum_{j=0}^{i-1} J_jP_{j-i+k} \equiv -\sum_{j=i}^{N} J_jP_{j-i+k}, \qquad i = 1, \ldots, N. \tag{2.10}$$

[1] Thus in particular for $0 \leqslant k \leqslant N$ the formula (2.7) takes the form

$$\sum_{i=k}^{N} J_iP_{i-k} = 0.$$

The above equivalence is immediately apparent form the equality (2.7) and will be used in subsection 2.3 for reconstructing the known [1] multi-Hamiltonian structure of the cKdV hierarchy.

For $N = 1$ the cKdV hierarchy reduces to the KdV hierarchy and for $N = 2$ to the Dispersive Water Waves (DWW) hierarchy.

2.2 Algebraic recursion

To solve (2.5) for the coefficients P_i we need to integrate differential equations provided by this condition or equivalently by (2.7). Integrating (2.5) we obtain

$$\frac{1}{2}\mathcal{P}\mathcal{P}_{xx} - \frac{1}{4}\mathcal{P}_x^2 + \mathbb{Q}\mathcal{P}^2 = c(\lambda) \equiv -4\lambda^N, \tag{2.11}$$

where $c(\lambda)$ is a polynomial in λ with coefficient being constants of integration of differential equations provided by (2.5). Here, for convenience, we make the simplest possible choice $c(\lambda) \equiv -4\lambda^N$ so that $P_0 = 2$. Other choices lead to hierarchies (2.9) with members being linear combinations of symmetries originating from the simplest possible choice as described above and by a linear change of basis in the cKdV hierarchy we can always choose the polynomial constant $c(\lambda)$ as in (2.11).

We will now attempt to solve (2.11) recursively for the coefficients P_i. Let us start by defining the following auxiliary differential (symmetric) bi-linear operators:

$$\mathcal{J}(f,g) := \sum_{i=0}^{N} \mathcal{J}_i(f,g)\lambda^i, \qquad \mathcal{J}_i(f,g) := -\frac{1}{16}\delta_{i0}\left(f_{xx}g + fg_{xx} - f_x g_x\right) - \frac{1}{4}u_i fg. \tag{2.12}$$

Thus

$$\mathcal{J}(f,g) \equiv -\frac{1}{16}(f_{xx}g + fg_{xx} - f_x g_x) - \frac{1}{4}\mathbb{Q}fg. \tag{2.13}$$

The bi-linear operators (2.12) and the linear operators (2.3) are related by the following useful formulas:

$$\left[\mathcal{J}_i(f,g)\right]_x = -\frac{1}{4}\left(fJ_i g + gJ_i f\right) \tag{2.14}$$

and

$$\left[\mathcal{J}(f,f)\right]_x = -\frac{1}{2}fJf.$$

Using the above b-linear operators, the equation (2.11) can be written in the equivalent form:

$$\mathcal{J}(\mathcal{P},\mathcal{P}) = \lambda^N. \tag{2.15}$$

Now, since

$$\mathcal{J}(\mathcal{P},\mathcal{P}) \equiv \sum_{i=0}^{N}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{J}_i(P_j,P_k)\lambda^{i-j-k} \equiv \sum_{k=-\infty}^{N}\sum_{i=0}^{N}\sum_{j=0}^{i-k}\mathcal{J}_i(P_j,P_{i-j-k})\lambda^k,$$

by (2.12) and since $P_0 = 2$ we find that $\mathcal{J}_N(P_0,P_0) = 1$ and thus, from the above equation, it follows that

$$\sum_{i=0}^{N}\sum_{j=0}^{i-k}\mathcal{J}_i(P_j,P_{i-j-k}) = 0 \qquad \text{for} \qquad k < N. \tag{2.16}$$

In [1] it was noticed that we can always solve (2.11) for coefficients P_i in terms of previously calculated (differential) expressions in $u_0,\ldots,u_{N-1}$. The following proposition provides us with a compact formula for that.

Proposition 2.1. *The coefficients P_i of the series* (2.6) *satisfy the following recursive formula*

$$P_k = -\sum_{j=1}^{k-1}\mathcal{J}_N(P_j,P_{k-j}) - \sum_{i=0}^{N-1}\sum_{j=0}^{i+k-N}\mathcal{J}_i(P_j,P_{i-j+k-N}), \qquad k = 1,2,\ldots. \tag{2.17}$$

Note that (2.17) is indeed of a recursive form as the right hand side contains P_i only up to P_{k-1}. Note also that this formula is purely differential-algebraic.

Proof. The recursion (2.17) is a consequence of the formula (2.16), which can be rewritten in the form

$$\sum_{i=0}^{N}\sum_{j=0}^{i+k-N}\mathcal{J}_i(P_j,P_{i-j+k-N})=0, \qquad \text{where} \quad k\geqslant 1. \tag{2.18}$$

For fixed k (2.18) involves only coefficients P_i for $0\leqslant i\leqslant k$ and P_k can be found only in the terms $\mathcal{J}_N(P_0,P_k)=\frac{1}{2}P_k$. Thus, solving for P_k we find the recursion (2.17). □

Explicitly, (2.17) can be written as

$$P_k=\frac{1}{4}\left[-\sum_{j=1}^{k-1}P_{k-j}P_j+\sum_{j=0}^{k-N}\left(\frac{1}{2}P_{k-j-N}(P_j)_{xx}-\frac{1}{4}(P_{k-j-N})_x(P_j)_x\right)+\sum_{i=0}^{N-1}\sum_{j=0}^{i+k-N}u_iP_{i-j+k-N}P_j\right], \tag{2.19}$$

where $k=1,2,\ldots$. The formula (2.17) (or (2.19)) was not present in literature before and it provides us with an effective way of calculating higher coefficients P_i from the lower ones without any need of integration. The functions P_i turn out to be components of cosymmetries of the cKdV hierarchy (2.9), see the formula (2.20) below.

2.3 Multi-Hamiltonian structure

The evolution equations from the N-component cKdV hierarchy (2.9) are multi-Hamiltonian with respect to $N+1$ mutually compatible Hamiltonian operators $\mathbb{B}_m$: [1, 25]

$$\boldsymbol{u}_{t_r}=\boldsymbol{K}_r\equiv\mathbb{B}_0\boldsymbol{\gamma}_r=\ldots=\mathbb{B}_m\boldsymbol{\gamma}_{r-m}=\ldots=\mathbb{B}_N\boldsymbol{\gamma}_{r-N}, \qquad m=0,1,\ldots,N, \tag{2.20}$$

where $\boldsymbol{\gamma}_r=(P_r,\ldots,P_{r+N-1})^T$ are cosymmetries of the hierarchy with P_i given by (2.17) or (2.19). The first and the last Hamiltonian structure (that is with respect to the Hamiltonian operators $\mathbb{B}_0$ and $\mathbb{B}_N$) are direct consequences of the two representations the hierarchy expressed in (2.10). The remaining Hamiltonian structures with respect to the Hamiltonian operators $\mathbb{B}_m$ can be constructed taking m first equations from the structure with respect to the operator $\mathbb{B}_N$ and $N-m$ last equations from the structure with respect to the operator $\mathbb{B}_0$. So, the Hamiltonian operators $\mathbb{B}_m$, for $m=0,1,\ldots N$, act on an arbitrary covector field $\boldsymbol{\eta}=(\eta_1,\ldots,\eta_N)^T$ as

$$\left(\mathbb{B}_m\boldsymbol{\eta}\right)_j=\sum_{i=0}^{j-1}J_i\eta_{i-j+m+1} \qquad \text{for} \qquad 1\leqslant j\leqslant m,$$
$$\left(\mathbb{B}_m\boldsymbol{\eta}\right)_j=-\sum_{i=j}^{N}J_i\eta_{i-j+m+1} \qquad \text{for} \qquad m+1\leqslant j\leqslant N,$$

Thus, the operators $\mathbb{B}_m$ have the explicit form:

$$\mathbb{B}_m=\left(\begin{array}{ccc|ccc} & & J_0 & & & \\ & \iddots & \vdots & & 0 & \\ J_0 & \cdots & J_{m-1} & & & \\ \hline & & & -J_{m+1} & \cdots & -J_N \\ & 0 & & \vdots & \iddots & \\ & & & -J_N & & \end{array}\right), \qquad m=0,\ldots,N.$$

Any two consecutive Hamiltonian operators $\mathbb{B}_m$ define (the same) hereditary recursion operator $\mathbb{R}$ through

$$\mathbb{R}:=\mathbb{B}_{m+1}\mathbb{B}_m^{-1}, \qquad m=0,1,\ldots,N-1,$$

given explicitly by

$$\mathbb{R} = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & -J_0 J_N^{-1} \\ \hline 1 & & & -J_1 J_N^{-1} \\ & \ddots & & \vdots \\ & & 1 & -J_{N-1} J_N^{-1} \end{array}\right), \tag{2.21}$$

so that one can generate all the vector fields of cKdV hierarchy and their cosymmetries with the help of the recursion operator $\mathbb{R}$ through

$$\boldsymbol{K}_{r+1} = \mathbb{R}^r \boldsymbol{K}_1 \qquad \text{and} \qquad \boldsymbol{\gamma}_{r+1-N} = (\mathbb{R}^\dagger)^r \boldsymbol{\gamma}_{1-N}, \qquad r = 1, 2, \ldots,$$

where $\boldsymbol{K}_1 = \boldsymbol{u}_x$ and $\boldsymbol{\gamma}_{1-N} = (0, \ldots, P_0)^T$. Let us however point out that the above method of constructing the hierarchy requires integrating the nonlocal operator (2.21) while our formula (2.17) gives us an explicit (although recursive) form of all P_k that are obtained by purely differential operations.

In what follows we will need two propositions (Proposition 2.2 and Proposition 2.3) that characterize the kernels of the Hamiltonian operators $\mathbb{B}_m$. In order to formulate and prove these propositions we need to define the following functions:

$$f_{k,m}(\boldsymbol{\xi}) := \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \mathcal{J}_i(\xi_{m+j-k}, \xi_{m+i-j+1}), \qquad 1 \leqslant k \leqslant m, \tag{2.22a}$$

$$g_{k,m}(\boldsymbol{\xi}) := -2 \sum_{i=k}^{N} \sum_{j=k}^{i} \mathcal{J}_i(P_{j-k}, \xi_{m+i-j+1}), \qquad m+1 \leqslant k \leqslant N, \tag{2.22b}$$

$$\tilde{g}_{k,m}(\boldsymbol{\xi},) := \sum_{i=k}^{N} \sum_{j=k}^{i} \mathcal{J}_i(\xi_{m+j-k+1}, \xi_{m+i-j+1}), \qquad m+1 \leqslant k \leqslant N, \tag{2.22c}$$

where $m \in \{0, \ldots, N\}$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)^T$ with $\xi_i = \xi_i[\boldsymbol{u}]$ is an arbitrary covector.

Lemma 2.1. *We have:*

$$[f_{k,m}(\boldsymbol{\xi})]_x = -\frac{1}{2} \sum_{j=1}^{k} \xi_{m+j-k} \big(\mathbb{B}_m \boldsymbol{\xi}\big)_j, \tag{2.23a}$$

$$[g_{k,m}(\boldsymbol{\xi})]_x = -\frac{1}{2} \sum_{j=k}^{N} P_{j-k} \big(\mathbb{B}_m \boldsymbol{\xi}\big)_j, \tag{2.23b}$$

$$[\tilde{g}_{k,m}(\boldsymbol{\xi})]_x = \frac{1}{2} \sum_{j=k}^{N} \xi_{m+j-k+1} \big(\mathbb{B}_m \boldsymbol{\xi}\big)_j. \tag{2.23c}$$

Proof. Differentiating (2.22a) and (2.22c) with respect to x and using the relation (2.14) we see that

$$\begin{aligned} [f_{k,m}(\boldsymbol{\xi})]_x &= -\frac{1}{4} \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \big[\xi_{m+j-k} J_i \xi_{m+i-j+1} + \xi_{m+i-j+1} J_i \xi_{m+j-k}\big] \\ &\equiv -\frac{1}{2} \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \xi_{m+j-k} J_i \xi_{m+i-j+1} = -\frac{1}{2} \sum_{j=1}^{k} \xi_{m+j-k} \sum_{i=0}^{j-1} J_i \xi_{m+i-j+1} \end{aligned}$$

and

$$\begin{aligned} [\tilde{g}_{k,m}(\boldsymbol{\xi})]_x &= -\frac{1}{4} \sum_{i=k}^{N} \sum_{j=k}^{i} \big[\xi_{m+j-k+1} J_i \xi_{m+i-j+1} + \xi_{m+i-j+1} J_i \xi_{m+j-k+1}\big] \\ &\equiv -\frac{1}{2} \sum_{i=k}^{N} \sum_{j=k}^{i} \xi_{m+j-k+1} J_i \xi_{m+i-j+1} = -\frac{1}{2} \sum_{j=k}^{N} \xi_{m+j-k+1} \sum_{i=j}^{N} J_i \xi_{m+i-j+1}. \end{aligned}$$

Hence, we obtain (2.23a) and (2.23c). Differentiating (2.22b) we obtain

$$[g_{k,m}(\boldsymbol{\xi})]_x = \frac{1}{2}\sum_{i=k}^{N}\sum_{j=k}^{i}\left[\xi_{m+i-j+1}J_iP_{j-k} + P_{j-k}J_i\xi_{m+i-j+1}\right]$$
$$= \frac{1}{2}\sum_{i=k}^{N}\sum_{j=k}^{i}P_{j-k}J_i\xi_{m+i-j+1} = \frac{1}{2}\sum_{j=k}^{N}P_{j-k}\sum_{i=j}^{N}J_i\xi_{m+i-j+1}$$

as

$$\sum_{i=k}^{N}\sum_{j=k}^{i}\xi_{m+i-j+1}J_iP_{j-k} \equiv \sum_{i=k}^{N}\sum_{j=k}^{i}\xi_{m+j-k+1}J_iP_{i-j} \equiv \sum_{j=k}^{N}\xi_{m+j-k+1}\sum_{i=j}^{N}J_iP_{i-j} = 0,$$

where the last equality is a consequence of (2.7) since here $j \geqslant 0$. Hence, (2.23b) follows. □

The following proposition describes the form of kernels of all Hamiltonian operators $\mathbb{B}_m$.

Proposition 2.2. *For fixed* $m \in \{0, 1, \ldots, N\}$, $\boldsymbol{\xi} \in \ker \mathbb{B}_m$ *(that is* $\mathbb{B}_m\boldsymbol{\xi} = 0$*) if and only if*

$$f_{k,m}(\boldsymbol{\xi}) = c_k \qquad \textit{for} \qquad 1 \leqslant k \leqslant m, \tag{2.24a}$$
$$g_{k,m}(\boldsymbol{\xi}) = c_k \qquad \textit{for} \qquad m+1 \leqslant k \leqslant N, \tag{2.24b}$$

where $c_1, \ldots, c_N$ *are arbitrary constants.*

Proof. Let us assume that the conditions (2.24) hold. Differentiating (2.24a) and using (2.23a) we obtain the system

$$[f_{k,m}(\boldsymbol{\xi})]_x \equiv -\frac{1}{2}\sum_{j=1}^{k}\xi_{m+j-k}\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j = 0, \qquad 1 \leqslant k \leqslant m, \tag{2.25}$$

which recursively implies that

$$\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j = 0 \qquad \text{for} \qquad 1 \leqslant j \leqslant m.$$

Notice that the implication is correct since the formula (2.25) can be interpreted as the matrix product of the triangular matrix ξ_{m+j-k} with the constant non-zero diagonal term ξ_m with the vector $\mathbb{B}_m\boldsymbol{\xi}$. For $m \geqslant 1$ always $\xi_m \neq 0$, which follows from the condition: $J_0\xi_m = 0$, required by the fact that $\boldsymbol{\xi} \in \ker \mathbb{B}_m$.

Next, differentiating (2.24b) and using (2.23b) we have

$$[g_{k,m}(\boldsymbol{\xi})]_x = -\frac{1}{2}\sum_{j=k}^{N}P_{j-k}\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j = 0, \qquad m+1 \leqslant k \leqslant N, \tag{2.26}$$

from which

$$\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j = 0 \qquad \text{for} \qquad m+1 \leqslant j \leqslant N.$$

Thus, the conditions (2.24) imply that $\mathbb{B}_m\boldsymbol{\xi} = 0$. The reverse implication is a matter of straightforward integration of (2.25) and (2.26). □

Later we will also need an alternative description of kernels of $\mathbb{B}_m$, contained in the following proposition.

Proposition 2.3. *Let us fix* $m \in \{0, 1, \ldots, N\}$ *and a natural* $n > 0$ *such that* $n + m \leqslant N - 2$*. Then* $\boldsymbol{\xi} \in \ker \mathbb{B}_m$ *(that is* $\mathbb{B}_m\boldsymbol{\xi} = 0$*) if and only if*

$$f_{k,m}(\boldsymbol{\xi}) = c_k \qquad \textit{for} \qquad 1 \leqslant k \leqslant m, \tag{2.27a}$$
$$g_{k,m}(\boldsymbol{\xi}) + \tilde{g}_{k+n+1,m}(\boldsymbol{\xi}) = c_k \qquad \textit{for} \qquad m+1 \leqslant k \leqslant N-n-1, \tag{2.27b}$$
$$g_{k,m}(\boldsymbol{\xi}) = c_k \qquad \textit{for} \qquad N-n \leqslant k \leqslant N, \tag{2.27c}$$

where $c_1, \ldots, c_N$ *are arbitrary constants.*

Proof. Just like in the previous proof, one can see that (2.27a) and (2.27c) imply that

$$\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j = 0 \qquad \text{for} \qquad 1 \leqslant j \leqslant m \quad \text{and} \quad N-n \leqslant j \leqslant N. \tag{2.28}$$

Finally, differentiating (2.27b) and taking into account (2.28),

$$[g_{k,m}(\boldsymbol{\xi}) + \tilde{g}_{k+n+1,m}(\boldsymbol{\xi})]_x = -\frac{1}{2}\sum_{j=k}^{N-n-1} P_{j-k}\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j + \frac{1}{2}\sum_{j=k+n+1}^{N-n-1} \xi_{m+j-k-n}\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j = 0, \tag{2.29}$$

where $m+1 \leqslant k \leqslant N-n-1$. Thus, (2.29) recursively implies that

$$\left(\mathbb{B}_m\boldsymbol{\xi}\right)_j = 0 \qquad \text{for} \qquad m+1 \leqslant j \leqslant N-n-1. \tag{2.30}$$

Hence, collecting together (2.28) and (2.30) we actually see that from the conditions (2.27) it follows that $\mathbb{B}_m\boldsymbol{\xi} = 0$. The reverse implication is straightforward. □

Remark 2.1. Let us notice that the characterizations of the kernels of the Hamiltonian operators $\mathbb{B}_m$, as in the above propositions, are not the only possible ones. For instance, Proposition 2.2 would still be correct if the functions $g_{k,m}$ in (2.24b) were entirely replaced by $\tilde{g}_{k,m}$, as defined in (2.22c). Choices made in Propositions 2.2 and 2.3 are dictated by later needs.

2.4 Lax representation

Introducing the vector eigenfunction $\Psi = (\psi, \psi_x)^T$ we can rewrite the linear problems (2.1) for the cKdV hierarchy (2.9) in the form

$$\Psi_{t_k} = \mathbb{V}_k\Psi, \qquad k = 1, 2, 3, \ldots, \tag{2.31}$$

where $t_1 \equiv x$ and

$$\mathbb{V}_1 = \begin{pmatrix} 0 & 1 \\ -\mathbb{Q} & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{V}_k = \begin{pmatrix} -\frac{1}{4}\left(\mathbb{P}_k\right)_x & \frac{1}{2}\mathbb{P}_k \\ -\frac{1}{4}\left(\mathbb{P}_k\right)_{xx} - \frac{1}{2}\mathbb{Q}\mathbb{P}_k & \frac{1}{4}\left(\mathbb{P}_k\right)_x \end{pmatrix}, \qquad k = 2, 3, \ldots. \tag{2.32}$$

see [1]. Then, by the compatibility conditions $(\Psi_x)_{t_k} = (\Psi_{t_k})_x$ and $(\Psi_{t_k})_{t_s} = (\Psi_{t_k})_{t_s}$ we obtain, respectively, the Lax equations

$$\frac{d}{dt_k}\mathbb{V}_1 - \frac{d}{dx}\mathbb{V}_k + [\mathbb{V}_1, \mathbb{V}_k] = 0, \qquad k = 1, 2, \ldots \tag{2.33}$$

and the zero-curvature equations

$$\frac{d}{dt_k}\mathbb{V}_s - \frac{d}{dt_s}\mathbb{V}_k + [\mathbb{V}_s, \mathbb{V}_k] = 0, \qquad s, k = 2, 3, \ldots. \tag{2.34}$$

Thus, $\mathbb{V}_1$ can be considered as the Lax matrix of the cKdV hierarchy, the matrices $\mathbb{V}_k$ for $k > 1$ play the role of auxiliary matrices while the hierarchy itself can be obtained by the matrix Lax equations (2.33). The zero-curvature equations (2.34) are differential consequences of the hierarchy.

3 Stationary cKdV systems

In this section we consider (see also the special case considered in [13]) stationary cKdV systems. A stationary cKdV system is a system that originates by restricting the (infinite) cKdV hierarchy (2.9) to one of its stationary manifolds. We will then show that the resulting finite-dimensional integrable system can be in a very natural way associated with an appropriate Stäckel system. Actually, due to the fact that the N-component cKdV hierarchy is $(N+1)$-hamiltonian, see (2.20), we can perform this association on $N+1$ different ways.

The $(n+1)$-th stationary flow of the cKdV hierarchy (2.9) is determined by the following condition:

$$\boldsymbol{u}_{t_{n+1}} = 0 \qquad \text{or equivalently} \qquad \boldsymbol{K}_{n+1} = 0, \tag{3.1}$$

which by (2.10) takes the form of a system of N differential equations:

$$(\boldsymbol{K}_{n+1})_j \equiv \sum_{i=0}^{j-1} J_i P_{i-j+n+1} \equiv -\sum_{i=j}^{N} J_i P_{i-j+n+1} = 0, \qquad j = 1, \ldots, N. \tag{3.2}$$

The stationary condition (3.1) provides a constraint (or rather a system of constraints) on the infinite-dimensional (functional) manifold $\mathcal{F}$, on which the cKdV hierarchy is defined, reducing it to the finite-dimensional submanifold, n-th stationary manifold:

$$\mathcal{M}_n = \{[\boldsymbol{u}] \in \mathcal{F} \,|\, \boldsymbol{K}_{n+1} = 0\}.$$

Due to complete integrability of the cKdV hierarchy, the constraints provided by (3.1) are invariant with respect to all the flows of the hierarchy. As a result, the infinite hierarchy (2.9) reduces to the finite system (3.3) described in the following definition.

Definition 3.1. The n-th stationary cKdV system is the system consisting of the first n evolution equations from the cKdV hierarchy (2.9) together with its $(n+1)$-th stationary flow:

$$\boldsymbol{u}_{t_1} = \boldsymbol{K}_1, \qquad \boldsymbol{u}_{t_2} = \boldsymbol{K}_2, \qquad \ldots, \qquad \boldsymbol{u}_{t_n} = \boldsymbol{K}_n, \qquad \boldsymbol{K}_{n+1} = 0. \tag{3.3}$$

From the recursive formula (2.19) one can observe that the cumulative differential order (i.e. the sum of differential orders of all components) of the vector field $\boldsymbol{K}_k$ increases by two as k increases by one. Thus, the cumulative order of $(n+1)$-th vector field $\boldsymbol{K}_{n+1}$ is equal $N+2n$, which means that the vector field $\boldsymbol{K}_{n+1}$ depends on $2(N+n)$ jet variables. Since the stationary condition (3.2) provides N independent constraints it follows that the stationary manifold $\mathcal{M}_n$ is $(2n+N)$-dimensional.

From the integrability of the cKdV hierarchy (2.9) it follows that the manifold $\mathcal{M}_n$ is invariant with respect to all the flows of the hierarchy and thus all the vector fields $\boldsymbol{K}_r$ in (3.3) are tangent to $\mathcal{M}_n$. They still pairwise commute since they commute on the ambient space $\mathcal{F}$. Note that also the higher vector fields $\boldsymbol{K}_{n+2}, \boldsymbol{K}_{n+3}, \ldots$ properly reduce to $\mathcal{M}_n$, however, we do not study these reductions in this article.

The system (3.3) is the main object of our study. In [13] the authors studied the particular case of (3.3) for $N = 1$, that is the stationary KdV system.

3.1 Hamiltonian foliations of $\mathcal{M}_n$

The (differential) order of the system of differential constraints (3.2) can be lowered by integrating them with respect to the spatial variable x. This procedure provides us with a system of differential constraints parameterized by N integration constants. It turns out that it can be done on $N+1$ different ways, due to the $(N+1)$ Hamiltonian structures of cKdV hierarchy (2.20). In particular, for the m-th Hamiltonian representation $\mathbb{B}_m \boldsymbol{\gamma}_{n+1-m} = 0$ of the $(n+1)$-th stationary cKdV flow (3.1) we can see that on the stationary manifold $\mathcal{M}_n$ the covector $\boldsymbol{\gamma}_{n+1-m}$ belongs to the kernel of the respective Hamiltonian operator $\mathbb{B}_m$. As result, the 'integrated' constraints with respect to the m-th Hamiltonian structure can be obtained requiring that $\boldsymbol{\gamma}_{n+1-m}$ fulfill the conditions from Proposition 2.2 or Proposition 2.3.

Let us, by setting $\boldsymbol{\xi} = \boldsymbol{\gamma}_{n+1-m}$, where $\xi_i = P_{n-m+i}$, in (2.22), define the following auxiliary functions:

$$f_k := f_{k,m}(\boldsymbol{\gamma}_{n+1-m}) \equiv \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \mathcal{J}_i(P_{j+n-k}, P_{i-j+n+1}), \tag{3.4a}$$

$$g_k := g_{k,m}(\boldsymbol{\gamma}_{n+1-m}) \equiv -2 \sum_{i=k}^{N} \sum_{j=k}^{i} \mathcal{J}_i(P_{j-k}, P_{i-j+n+1}), \tag{3.4b}$$

$$\tilde{g}_k := \tilde{g}_{k,m}(\boldsymbol{\gamma}_{n+1-m}) \equiv \sum_{i=k}^{N} \sum_{j=k}^{i} \mathcal{J}_i(P_{j+n-k+1}, P_{i-j+n+1}), \tag{3.4c}$$

where in each case $1 \leqslant k \leqslant N$. Observe that they do not depend on m.

Thus, by Proposition 2.2, by setting $\boldsymbol{\xi} = \boldsymbol{\gamma}_{n+1-m}$ in (2.24) we find the following integrated form of the $(n+1)$-th stationary cKdV flow (3.1):

$$f_k = c_k \qquad \text{for} \qquad 1 \leqslant k \leqslant m, \tag{3.5a}$$

$$g_k = c_k \qquad \text{for} \qquad m+1 \leqslant k \leqslant N, \tag{3.5b}$$

where $c_1, \ldots, c_N$ are (arbitrary) integration constants. Moreover, for cases that $n + m < N - 1$, by Proposition 2.3, setting $\boldsymbol{\xi} = \boldsymbol{\gamma}_{n+1-m}$ in (2.27) yields the following alternative integrated form of the $(n+1)$-th stationary cKdV flow (3.1):

$$f_k = c_k \qquad \text{for} \qquad 1 \leqslant k \leqslant m, \tag{3.6a}$$

$$g_k + \tilde{g}_{k+n+1} = c_k \qquad \text{for} \qquad m+1 \leqslant k \leqslant N-n-1, \tag{3.6b}$$

$$g_k = c_k \qquad \text{for} \qquad N-n \leqslant k \leqslant N, \tag{3.6c}$$

Therefore, for each $m \in \{0, 1, \ldots, N\}$ the above relations define a $2n$-dimensional foliation of the stationary manifold $\mathcal{M}_n$, parameterized by the vector $\boldsymbol{c} \equiv (c_1, \ldots, c_N)$. The leaves of this foliation are given by

$$\mathcal{M}_{n,m}^{\mathbf{c}} := \{[\boldsymbol{u}] \in \mathcal{M}_n \,|\, \text{s.t. (3.5) for } n+m \geqslant N-1 \text{ or (3.6) for } n+m < N-1\}\,, \tag{3.7}$$

so that for each m:

$$\mathcal{M}_n \equiv \bigcup_{\boldsymbol{c} \in \mathbb{R}^N} \mathcal{M}_{n,m}^{\mathbf{c}}. \tag{3.8}$$

We will refer to this foliation as *Hamiltonian foliation* of $\mathcal{M}_n$. The case $n + m \geqslant N - 1$ will be referred to as the *generic* case while the case $n + m < N - 1$ we will call the *non-generic* case.

Remark 3.1. The Hamiltonian foliation (3.8) of the n-th stationary manifold $\mathcal{M}_n$ could be defined in a simpler way, through the leaves

$$\mathcal{M}_{n,m}^{\mathbf{c}} := \{[\boldsymbol{u}] \in \mathcal{M}_n \,|\, \text{s.t. (3.5)}\}\,,$$

given only by the relations (3.5), i.e. without introducing the non-generic case. We will however use the definition (3.7) which is motivated by later needs.

3.2 Stäckel foliations of $\mathcal{M}_n$

The stationary manifold $\mathcal{M}_n$ can also be foliated in a way that allows for representing a given cKdV stationary system as a Stäckel system defined on leaves of this foliation. We will therefore call this foliation (see its definition below) of $\mathcal{M}_n$ the *Stäckel foliation*. In fact, we will construct $N + 1$ different Stäckel foliations of $\mathcal{M}_n$, one for each choice of $m \in \{0, \ldots, N\}$.

We start by observing that the $(n+1)$-th stationary flow (3.1) of the cKdV hierarchy can be written as

$$\mathbb{Q}_{t_{n+1}} = (\mathbb{P}_{n+1})_x \mathbb{Q} + \frac{1}{2} \mathbb{P}_{n+1} \mathbb{Q}_x + \frac{1}{4} (\mathbb{P}_{n+1})_{3x} \equiv J \mathbb{P}_{n+1} = 0. \tag{3.9}$$

Note that this equation contains not only the stationary flow $K_{n+1} = 0$ but also the first n equations of the infinite recursion (2.5) on P_k. The stationary condition (3.9) can be integrated once to the form:

$$-\frac{1}{8}\mathbb{P}_{n+1}(\mathbb{P}_{n+1})_{xx} + \frac{1}{16}(\mathbb{P}_{n+1})_x^2 - \frac{1}{4}\mathbb{Q}\mathbb{P}_{n+1}^2 = C(\lambda), \tag{3.10}$$

or in our shorthand notation, using (2.13), as

$$\mathcal{J}(\mathbb{P}_{n+1}, \mathbb{P}_{n+1}) = C(\lambda), \tag{3.11}$$

where $C(\lambda)$ is an appropriate polynomial in λ with coefficients being integration constants that follow from the next proposition. Below we investigate the left hand side of (3.11) more thoroughly.

Proposition 3.1. *The left-hand side of* (3.11)*, or* (3.10)*, takes the following explicit form*

$$\mathcal{J}(\mathbb{P}_{n+1}, \mathbb{P}_{n+1}) \equiv \lambda^{2n+N} + \sum_{k=0}^{n+N-1} h_k \lambda^k, \tag{3.12}$$

where the coefficients h_k are differential functions of $\boldsymbol{u}$ given by

$$h_k = \sum_{i=0}^{N} \sum_{j=i}^{k} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}). \tag{3.13}$$

For $n + N \le k < 2n + N$ the coefficients h_k in (3.13) vanish and $h_{2n+N} = 1$.[2]

Proof. For $k = n + 1$ (2.4) has the form

$$\mathbb{P}_{n+1} \equiv [\lambda^n \mathcal{P}]_{\geqslant 0} = \sum_{i=0}^{n} P_{n-i}\lambda^i.$$

Thus,

$$\begin{aligned}\mathcal{J}(\mathbb{P}_{n+1}, \mathbb{P}_{n+1}) &= \sum_{i=0}^{N}\sum_{j=0}^{n}\sum_{k=0}^{n} \mathcal{J}_i(P_{n-j}, P_{n-k})\lambda^{i+j+k} = \sum_{i=0}^{N}\sum_{j=i}^{n+i}\sum_{k=j}^{n+j} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})\lambda^k \\ &\equiv \sum_{k=0}^{2n+N}\sum_{i=0}^{N}\sum_{j=i}^{k} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})\lambda^k = \sum_{k=0}^{2n+N} h_k\lambda^k,\end{aligned}$$

Taking into account that $P_{n-k-j} = 0$ for $j < k - n$ and $P_{n+i-j} = 0$ for $j > n + i$, we obtain that for $k = 2n + N$

$$h_{2n+N} = \sum_{i=0}^{N}\sum_{j=i}^{2n+N} \mathcal{J}_i(P_{j-n-N}, P_{n+i-j}) = \mathcal{J}_N(P_0, P_0) = 1,$$

while for $n + N \leqslant k \leqslant 2n + N - 1$ we obtain

$$h_k = \sum_{i=0}^{N}\sum_{j=k-n}^{n+i} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) = \sum_{i=0}^{N}\sum_{j=0}^{i+2n-k} \mathcal{J}_i(P_j, P_{i-j+2n-k}) = 0,$$

were we used the equality (2.16). The remaining coefficients h_k are non-vanishing. □

From Proposition 3.1 it follows that

$$C(\lambda) \equiv \lambda^{2n+N} + \sum_{k=0}^{n+N-1} \varepsilon_k\lambda^k,$$

[2] It is worth here to compare (3.11) with $\mathcal{J}(\lambda^n\mathcal{P}, \lambda^n\mathcal{P}) = \lambda^{2n+N}$, which follows from (2.15).

where ε_k are some integration constants. Due to (3.11) the functions (3.13) are integrals of motion of all the flows of the n-th stationary system (3.3). The integration constants ε_k are thus the values of integrals of motion h_k, depending on particular initial conditions.

In two lemmas below we analyze the functions h_k more closely.

Lemma 3.1. *For $k < N$:*

$$(h_k)_x = -\frac{1}{2}\sum_{j=0}^{k} P_{n-k+j}\sum_{i=0}^{j} J_i P_{n+i-j} \equiv -\frac{1}{2}\sum_{j=0}^{k} P_{n-k+j}(\boldsymbol{K}_{n+1})_{j+1}, \tag{3.14a}$$

and if $k \geqslant N$:

$$(h_k)_x = -\frac{1}{2}\sum_{j=0}^{N-1} P_{n-k+j}\sum_{i=j+1}^{N} J_i P_{n+i-j} \equiv -\frac{1}{2}\sum_{j=0}^{N-1} P_{n-k+j}(\boldsymbol{K}_{n+1})_{j+1}, \tag{3.14b}$$

moreover if additionally $k \geqslant n$:

$$(h_k)_x = -\frac{1}{2}\sum_{j=k-n}^{N-1} P_{n-k+j}\sum_{i=j+1}^{N} J_i P_{n+i-j} \equiv -\frac{1}{2}\sum_{j=k-n}^{N-1} P_{n-k+j}(\boldsymbol{K}_{n+1})_{j+1}. \tag{3.14c}$$

Proof. Differentiating (3.13) with respect to x and using the relation (2.14) one finds that

$$(h_k)_x = -\frac{1}{4}\sum_{i=0}^{N}\sum_{j=i}^{k}\left[P_{n+i-j}J_iP_{n-k+j} + P_{n-k+j}J_iP_{n+i-j}\right] \equiv -\frac{1}{2}\sum_{i=0}^{N}\sum_{j=i}^{k} P_{n-k+j}J_iP_{n+i-j}. \tag{3.15}$$

Thus, for $k < N$, using (2.10), we obtain (3.14a). For $k \geqslant N$ one finds that (3.15) takes the form

$$(h_k)_x \equiv -\frac{1}{2}\sum_{j=0}^{N-1} P_{n-k+j}\sum_{i=0}^{j} J_iP_{n+i-j} - \frac{1}{2}\sum_{j=N}^{k} P_{n-k+j}\sum_{i=0}^{N} J_iP_{n+i-j},$$

where the second term vanishes by the equality (2.7), now using (2.10) we get (3.14b). □

We will now prove that the manifold $\mathcal{M}_n$ can be reconstructed, on $N+1$ different ways, from appropriate subsets of the set of all functions h_k. This is the content of the next theorem.

Theorem 3.1. *Let us fix $m \in \{0, \ldots, N\}$. Then, the set of solutions of the system of equations*

$$h_k = c_{k+1} \qquad k = 0, \ldots, m-1 \tag{3.16a}$$

and

$$h_k = c_{k-n+1}, \qquad k = n+m, \ldots, n+N-1. \tag{3.16b}$$

where all c_k vary over $\mathbb{R}$, coincide with the stationary manifold $\mathcal{M}_n$.

When we fix the values of all c_k then the equations (3.16) define a particular leaf of a $2n$-dimensional foliation of the stationary manifold $\mathcal{M}_n$. This foliation is parameterized by the vector $\boldsymbol{c} \equiv (c_1, \ldots, c_N)$. Therefore, for each $m \in \{0, 1, \ldots, N\}$, we define

$$\bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m} := \{[\boldsymbol{u}] \in \mathcal{M}_n \,|\, \text{s.t. } (3.16)\}, \tag{3.17}$$

and then $\mathcal{M}_n$ is foliated into $\bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m}$:

$$\mathcal{M}_n \equiv \bigcup_{\boldsymbol{c}\in\mathbb{R}^N} \bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m}.$$

Note that foliations (3.17) for different m are transversal to each other. We will refer to the foliation (3.17) as m-th *Stäckel foliation* of $\mathcal{M}_n$. By construction, the leaves $\bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m}$ are invariant with respect to the evolution flows from the n-th stationary system (3.3). This means that the mutually commuting vector fields K_k from the n-th stationary cKdV system (3.3) are tangent to all the leaves $\bar{\mathcal{M}}^{\mathbf{c}}_{n,m}$.

Proof. We have to show that for each $m \in \{0, 1, \dots, N\}$ the constraints (3.16) imply the conditions (3.2). We will show it by differentiating (3.16) and using (3.14) taking into account the relation between k and N. We need to consider two cases.

Let us start with the generic case (i.e. $n + m \geqslant N$). By (3.16):

$$h_k = c_{k+1} \qquad \text{for} \qquad 0 \leqslant k \leqslant m-1, \tag{3.18a}$$

$$h_k = c_{k-n+1} \qquad \text{for} \qquad n+m \leqslant k \leqslant n+N-1. \tag{3.18b}$$

Differentiating (3.18a) and using (3.14a) we obtain the system

$$(h_k)_x \equiv -\frac{1}{2}\sum_{j=0}^{k} P_{n-k+j}(\boldsymbol{K}_{n+1})_{j+1} = 0, \qquad 0 \leqslant k \leqslant m-1,$$

which recursively implies that

$$(\boldsymbol{K}_{n+1})_j = 0 \qquad \text{for} \qquad 1 \leqslant j \leqslant m.$$

Similarly, differentiating (3.18b) and using (3.14c) we obtain

$$(h_k)_x \equiv -\frac{1}{2}\sum_{j=k-n}^{N-1} P_{n-k+j}(\boldsymbol{K}_{n+1})_{j+1} = 0, \qquad n+m \leqslant k \leqslant N+n-1,$$

from which it follows that

$$(\boldsymbol{K}_{n+1})_j = 0 \qquad \text{for} \qquad m+1 \leqslant j \leqslant N.$$

Thus, for $n + m \geqslant N$ the relations (3.16) imply all the conditions (3.2).

In the non-generic case $n + m < N$ we first rewrite (3.16) in the form:

$$h_k = c_{k+1} \qquad \text{for} \qquad 0 \leqslant k \leqslant m-1, \tag{3.19a}$$

$$h_k = c_{k-n+1} \qquad \text{for} \qquad N \leqslant k \leqslant n+N-1. \tag{3.19b}$$

$$h_k = c_{k-n+1} \qquad \text{for} \qquad n+m \leqslant k < N. \tag{3.19c}$$

The relations (3.19a) together with (3.14a), similarly as (3.18a), imply that

$$(\boldsymbol{K}_{n+1})_j = 0 \qquad \text{for} \qquad 1 \leqslant j \leqslant m.$$

Differentiating (3.19b) and using (3.14c) we get the system

$$(h_k)_x \equiv -\frac{1}{2}\sum_{j=k-n}^{N-1} P_{n-k+j}(\boldsymbol{K}_{n+1})_{j+1} = 0, \qquad N \leqslant k \leqslant N+n-1,$$

which now implies that

$$(\boldsymbol{K}_{n+1})_j = 0 \qquad \text{for} \qquad N-n+1 \leqslant j \leqslant N. \tag{3.20}$$

We obtain the remaining cases differentiating (3.19c) and using, this time, (3.14a):

$$(h_k)_x \equiv -\frac{1}{2}\sum_{j=k-n}^{k} P_{n-k+j}(\boldsymbol{K}_{n+1})_{j+1} = 0, \qquad n+m \leqslant k < N, \tag{3.21}$$

since $P_{n-k+j} \neq 0$ only for $n - k + j \geqslant 0$. Taking into account (3.20) the system (3.21) implies that

$$(\boldsymbol{K}_{n+1})_j = 0 \qquad \text{for} \qquad m+1 \leqslant j \leqslant N-n.$$

Thus, appropriately gathering the above cases we actually see that also for $n+m \geqslant N$ the relations (3.16) imply all the conditions in (3.2). $\square$

Lemma 3.2. *The coefficients h_k in* (3.13) *satisfy the following relations:*

$$h_k = f_{k+1} \qquad \textit{for} \qquad k < N, \tag{3.22a}$$

$$h_k = g_{k-n+1} \qquad \textit{for} \qquad k \geqslant \max(N-1, n) \tag{3.22b}$$

$$h_k = g_{k-n+1} + \tilde{g}_{k+2} \qquad \textit{for} \qquad n \leqslant k < N-1. \tag{3.22c}$$

For the proof see Appendix.

Notice that, by Lemma 3.2, in the generic case $n+m \geqslant N-1$, the set of constraints (3.16) takes the form:

$$f_k \equiv h_{k-1} = c_k \qquad \text{for} \qquad 1 \leqslant k \leqslant m,$$

$$g_k \equiv h_{k+n-1} = c_k \qquad \text{for} \qquad m+1 \leqslant k \leqslant N.$$

Thus we can see that in the generic case the set of constraints (3.16) is identical with the 'integrated' set of constraints (3.5) associated with m-th Hamiltonian representation of the $(n+1)$-th stationary cKdV flow (3.1). By the same lemma, in the non-generic case $n+m < N-1$, the set of constraints (3.16) takes the form:

$$h_{k-1} \equiv f_k = c_k \qquad \text{for} \qquad 1 \leqslant k \leqslant m,$$

$$h_{k+n-1} \equiv g_k + \tilde{g}_{k+n+1} = c_k \qquad \text{for} \qquad m+1 \leqslant k \leqslant N-n,$$

$$h_{k+n-1} \equiv g_k = c_k \qquad \text{for} \qquad N-n+1 \leqslant k \leqslant N,$$

which is equivalent with (3.6). The above results lead to the following theorem.

Theorem 3.2. *The Hamiltonian foliation $\mathcal{M}^c_{n,m}$ and the Stäckel foliation $\bar{\mathcal{M}}^c_{n,m}$ coincide.*

Thus, in the sequel, we will only use the notation $\mathcal{M}^c_{n,m}$ for leaves of this foliation.

3.3 Stationary cKdV system on the leaves $\mathcal{M}^c_{n,m}$.

The constraint (3.1), defining the n-th stationary manifold $\mathcal{M}_n$, can also be obtained by imposing an appropriate constraint on the Lax hierarchy (2.31). To see this, let us impose the following constraint

$$\Psi_{t_{n+1}} = \lambda^m \mu \Psi \tag{3.23a}$$

or equivalently

$$\mathbb{V}_{n+1}\Psi = \lambda^m \mu \Psi \tag{3.23b}$$

on the linear problems (2.31). The factor $\lambda^m \mu$ is chosen here in order to relate the stationary system (3.3) with an appropriate Stäckel system. The parameters μ and λ are spectral parameters of the linear problems (2.31) and (3.23), and they are assumed to be isospectral, i.e. independent of all evolution variables. Then, the constraint (3.23a) and the compatibility condition $(\Psi_{t_k})_{t_{n+1}} = (\Psi_{t_{n+1}})_{t_k}$ yields

$$\frac{d}{dt_{n+1}}\mathbb{V}_k = 0, \quad k = 1, 2, \ldots \tag{3.24}$$

which due to the form of $\mathbb{V}_1$ in (2.32) is equivalent to the $(n+1)$-th stationary flow of the cKdV hierarchy (3.1). As a consequence, the equations (2.33) and (2.34), after imposing (3.24), yield the following Lax representation of the stationary cKdV system (3.3) on $\mathcal{M}_n$:

$$\frac{d}{dt_k}\mathbb{V}_{n+1} = [\mathbb{V}_k, \mathbb{V}_{n+1}], \qquad k = 1, \ldots, n.$$

In consequence, the stationary cKdV system (3.3) on $\mathcal{M}^{\mathbf{c}}_{n,m}$ has the following Lax representation

$$\frac{d}{dt_k}\mathbb{V}^{(m)}_{n+1}=[\mathbb{V}^{(m)}_{k},\mathbb{V}^{(m)}_{n+1}],\qquad k=1,2,\dots,n, \tag{3.25}$$

where $\mathbb{V}^{(m)}_{k}$ originate by inserting the constraints (3.16) into $\mathbb{V}_k$ given by (2.32). Note that now it is the matrix $\mathbb{V}_{n+1}$ (and respectively $\mathbb{V}^{(m)}_{n+1}$) that plays the role of the Lax matrix of the stationary cKdV system on $\mathcal{M}_n$ (and, respectively, on $\mathcal{M}^{\mathbf{c}}_{n,m}$), while the matrices $\mathbb{V}_k$ (and $\mathbb{V}^{(m)}_{k}$, respectively), for $k=1,\dots,n$, play the role of auxiliary matrices.

Nontrivial solutions for the linear problem (3.23b) exist provided that the characteristic equation

$$\det\left(\mathbb{V}_{n+1}-\lambda^m\mu\mathbb{I}\right)=0 \tag{3.26}$$

is satisfied. Explicitly written, the characteristic equation (3.26) takes the form

$$-\frac{1}{8}\mathbb{P}_{n+1}(\mathbb{P}_{n+1})_{xx}+\frac{1}{16}(\mathbb{P}_{n+1})_x^2-\frac{1}{4}\mathbb{Q}\mathbb{P}_{n+1}^2=\lambda^{2m}\mu^2, \tag{3.27}$$

where the left-hand side coincides with with the 'integrated' form (3.10) of the $(n+1)$-th stationary flow (3.9). The equation (3.27) can be written in our shorthand notation as

$$\mathcal{J}(\mathbb{P}_{n+1},\mathbb{P}_{n+1})=\lambda^{2m}\mu^2. \tag{3.28}$$

By Proposition 3.1, the relation (3.27) (or equivalently (3.28)) attains the form of a spectral curve

$$\lambda^{2n+N}+\sum_{k=0}^{n+N-1}h_k\lambda^k=\lambda^{2m}\mu^2, \tag{3.29}$$

where h_k are given by (3.13). Since the functions h_k depend not only on x but also on all evolution parameters $t_1,\dots,t_n$, it follows from (3.29) that they are in fact integrals of motion of all the flows of the stationary cKdV system (3.3) on $\mathcal{M}_n$, as we mentioned earlier.

With each foliation $\mathcal{M}^{\mathbf{c}}_{n,m}$ (one for each $m\in\{1,\dots,N\}$) we now associate the curve (3.29) with appropriate h_k fixed by (3.16), that is a curve (depending on N parameters c_k)

$$\lambda^{2n+N}+\sum_{k=1}^{N-m}c_{m+k}\lambda^{n+m+k-1}+\sum_{k=1}^{n}H_k\lambda^{n+m-k}+\sum_{k=1}^{m}c_k\lambda^{k-1}=\lambda^{2m}\mu^2, \tag{3.30}$$

where we use the notation

$$H_i=h_{n+m-i},\qquad i=1,\dots,n,$$

that will prove to be useful later. Dividing (3.30) by λ^m we obtain

$$\lambda^{2n+N-m}+\sum_{k=1}^{N-m}c_{m+k}\lambda^{n+k-1}+\sum_{k=1}^{n}H_k\lambda^{n-k}+\sum_{k=1}^{m}c_k\lambda^{k-m-1}=\lambda^{m}\mu^2. \tag{3.31}$$

As we will prove in the subsequent sections, if we now treat the curve (3.31) as a separation curve for a particular Stäckel system of Benenti type, we will be able to construct a map between jet variables on $\mathcal{M}_n$ and the separation variables of this Stäckel system such that it preserves the dynamics (i.e. it maps the solutions of one systems onto solutions of the other system).

4 Stäckel systems of Benenti type and their Lax representation

4.1 Stäckel systems in separation variables

This section contains some basic information about Stäckel separable systems of Benenti type. Consider the separation curve in the form (cf. (3.31))

$$\sigma(\lambda) + \sum_{k=1}^{n} H_k \lambda^{n-k} = \lambda^m \mu^2, \qquad m \in \mathbb{Z}, \tag{4.1}$$

where $\sigma(\lambda)$ is a (Laurent) polynomial in the variable λ. The separable systems associated with (4.1) belong to the so-called Benenti subclass of Stäckel systems [4, 11]. Consider now the coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ on a phase space $M = T^*Q$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$ are coordinates on an n-dimensional configuration space Q and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ are the fibre (momenta) coordinates. Taking n copies of (4.1) at n points (λ_i, μ_i) we obtain the system

$$\sigma(\lambda_i) + \sum_{k=1}^{n} H_k \lambda_i^{n-k} = \lambda_i^m \mu_i^2, \qquad i = 1, \ldots, n \tag{4.2}$$

that is linear with respect to H_k. It can thus be easily solved with respect to these variables yielding n functions $H_k = H_k(\boldsymbol{\lambda}, \boldsymbol{\mu})$ on M. In result, we obtain n quadratic in momenta (Stäckel) Hamiltonians:

$$H_k = \frac{1}{2}\boldsymbol{\mu}^T A_k G_m \boldsymbol{\mu} + V_k, \qquad k = 1, \ldots, n, \tag{4.3}$$

on the phase space T^*Q, where G_m are treated as contravariant metrics on the configuration space Q. Explicitly

$$G_m = 2\,\mathrm{diag}\left(\frac{\lambda_1^m}{\Delta_1}, \ldots, \frac{\lambda_n^m}{\Delta_n}\right), \qquad \Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j).$$

Further, all A_k are $(1,1)$-Killing tensors for all the metrics G_m and the are given by

$$A_k = (-1)^{k+1} \mathrm{diag}\left(\frac{\partial s_k}{\partial \lambda_1}, \ldots, \frac{\partial s_k}{\partial \lambda_n}\right), \qquad k = 1, \ldots, n,$$

where s_k denotes the elementary symmetric polynomial in variables λ_i of degree k, e.g.

$$s_1 = \sum_i \lambda_i, \qquad s_2 = \sum_{i<j} \lambda_i \lambda_j, \qquad \ldots, \qquad s_n = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Notice that the Stäckel matrix S associated with the linear system (4.2) is the Vandermote matrix, $S_{ij} = \lambda_i^{n-j}$, see [6], with the determinant and its inverse given by

$$\det S = \prod_{i<j} (\lambda_j - \lambda_i), \qquad (S^{-1})_{ij} = \frac{(-1)^{i+1}}{\Delta_j} \frac{\partial s_i}{\partial \lambda_j}. \tag{4.4}$$

Each metric G_m in (4.3) can be generated through $G_m = L^m G_0$, where $L := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a special conformal Killing tensor [17]. Due to (4.4), the potential functions V_k in (4.3) are given by

$$V_k = (-1)^{k+1} \sum_{i=1}^{n} \frac{\partial s_k}{\partial \lambda_i} \frac{\sigma(\lambda_i)}{\Delta_i}, \qquad k = 1, \ldots, n. \tag{4.5}$$

By construction, the Hamiltonians (4.3) are in involution with respect to the Poisson bracket

$$\{f, g\} = \pi(df, dg), \qquad f, g \in C^\infty(M),$$

with the Poisson bi-vector $\pi = \sum_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}$ (thus, $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are Darboux coordinates for π). The time evolution of any observable $\xi \in C^\infty(M)$ with respect to the Hamiltonian H_k has the form $\xi_{t_k} = \{\xi, H_k\}$ and the Hamiltonian evolution equations are

$$\boldsymbol{\lambda}_{t_k} = \{\boldsymbol{\lambda}, H_k\} \equiv \frac{\partial H_k}{\partial \boldsymbol{\mu}}, \qquad \boldsymbol{\mu}_{t_k} = \{\boldsymbol{\mu}, H_k\} \equiv -\frac{\partial H_k}{\partial \boldsymbol{\lambda}}, \qquad k = 1, \dots, n. \tag{4.6}$$

By construction, the variables $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are separation variables for all the Stäckel Hamiltonians H_k in (4.3).

4.2 Stäckel systems in Viète coordinates

Apart from the separation coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ we will also work with Viète coordinates, defined by

$$q_i = (-1)^i s_i, \qquad p_i = -\sum_{k=1}^n \frac{\lambda_k^{n-i} \mu_k}{\Delta_k}, \qquad i = 1, \dots, n. \tag{4.7}$$

The transformation (4.7) between the separation coordinates and the Viète coordinates is a canonical transformation (since it is a point transformation) so that $\pi = \sum_{i=1}^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$. In Viète coordinates the Stäckel Hamiltonians (4.3) take the form

$$H_k = \frac{1}{2} \boldsymbol{p}^T A_k G_m \boldsymbol{p} + V_k, \qquad k = 1, \dots, n, \tag{4.8}$$

(where $\boldsymbol{p} = (p_1, \dots, p_n)^T$ and $\boldsymbol{q} = (q_1, \dots, q_n)^T$) and the respective Hamiltonian equations attain the form

$$\boldsymbol{q}_{t_k} = \{\boldsymbol{q}, H_k\} \equiv \frac{\partial H_k}{\partial \boldsymbol{p}}, \qquad \boldsymbol{p}_{t_k} = \{\boldsymbol{p}, H_k\} \equiv -\frac{\partial H_k}{\partial \boldsymbol{q}}, \qquad k = 1, \dots, n. \tag{4.9}$$

If $\sigma(\lambda) = \sum_i a_i \lambda^i$ the potential functions (4.5) are given by $V_k = \sum_i a_i \mathcal{V}_k^{(i)}$, where the so-called elementary separable potentials $\mathcal{V}_k^{(i)}$ can be explicitly constructed from the recursion formula [12]

$$\mathcal{V}^{(i)} = R^i \mathcal{V}^{(0)}, \qquad \mathcal{V}^{(i)} = \left(\mathcal{V}_1^{(i)}, \dots, \mathcal{V}_n^{(i)}\right)^T, \qquad \mathcal{V}^{(0)} = (0, \dots, 0, -1)^T,$$

where

$$R = \begin{pmatrix} -q_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -q_n & 0 & 0 & 0 \end{pmatrix}, \qquad R^{-1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{q_n} \\ 1 & 0 & 0 & \vdots \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -\frac{q_{n-1}}{q_n} \end{pmatrix}. \tag{4.10}$$

In Viète coordinates the metric G_0 has the form $(G_0)^{ij} = 2q_{i+j-n-1}$, where, for convenience, we set $q_0 = 1$ and $q_i = 0$ for $i < 0$ or $i > n$. The metrics for arbitrary m are given again by $G_m = L^m G_0$. An interesting fact is that in Viète coordinates the special conformal Killing tensor L has the matrix representation identical to R defined by (4.10) [11]. Further, the Killing tensors A_k, are in these coordinates given by

$$(A_k)^i_j = \begin{cases} q_{i-j+k-1} & \text{if} \quad i \leqslant j \quad \text{and} \quad k \leqslant j, \\ -q_{i-j+k-1} & \text{if} \quad i > j \quad \text{and} \quad k > j, \\ 0 & \text{otherwise}, \end{cases}$$

where $k = 1, \dots, n$. Notice that $(A_1)^i_j = \delta^i_j$ in any coordinate system.

Reversing the fiber part of the map (4.7) we find that

$$\mu_i = \sum_{k=1}^n (-1)^k \frac{\partial s_k}{\partial \lambda_i} p_k.$$

Hence, we can obtain the formula

$$\sum_{i=1}^{n}\frac{\lambda_i^k\mu_i}{\Delta_i}=-\sum_{j=1}^{n}\mathcal{V}_j^{(k)}p_j,\qquad k\in\mathbb{Z}, \tag{4.11}$$

that we will be useful later. It follows directly by (4.4) and the fact that the elementary potentials in separation coordinates are given by

$$\mathcal{V}_j^{(k)}=(-1)^{j+1}\sum_{i=1}^{n}\frac{\partial s_j}{\partial\lambda_i}\frac{\lambda_i^k}{\Delta_i},$$

which in turn is an immediate consequence of (4.5).

4.3 Lax representation of Stäckel systems

Let us now, following [7], present the Lax formulation of Stäckel systems of Benenti type. The results below are necessary for the proof of the main theorem of this paper, Theorem 5.1.

The Hamiltonian evolution equations (4.6) or (4.9), associated with the separation (spectral) curves (4.1), have the following (isospectral) Lax representation

$$\frac{d}{dt_k}\mathbb{L}=[\mathbb{U}_k,\mathbb{L}],\qquad k=1,\dots,n, \tag{4.12}$$

with $\mathbb{L}$ and $\mathbb{U}_k$ being 2×2 traceless matrices depending rationally on the spectral parameter λ. The Lax matrix $\mathbb{L}$ has the form[3]

$$\mathbb{L}=\begin{pmatrix}\mathfrak{v} & \mathfrak{u}\\ \mathfrak{w} & -\mathfrak{v}\end{pmatrix}, \tag{4.13}$$

where $\mathfrak{u}$ is a function on Q given in the separation coordinates $\boldsymbol{\lambda}$ by

$$\mathfrak{u}:=\prod_{k=1}^{n}(\lambda-\lambda_k)\equiv\lambda^n+\sum_{k=1}^{n}(-1)^k s_k\lambda^{n-k},$$

while $\mathfrak{v}$ and $\mathfrak{w}$ are functions on T^*Q given in the separation coordinates $(\boldsymbol{\lambda},\boldsymbol{\mu})$ by

$$\mathfrak{v}:=\sum_{k=1}^{n}(-1)^{k+1}\left[\sum_{i=1}^{n}\frac{\partial s_k}{\partial\lambda_i}\frac{\lambda_i^m\mu_i}{\Delta_i}\right]\lambda^{n-k}$$

and

$$\mathfrak{w}:=\frac{1}{\mathfrak{u}}\left[\lambda^m\left(\sigma(\lambda)+\sum_{k=1}^{n}H_k\lambda^{n-k}\right)-\mathfrak{v}^2\right]. \tag{4.14}$$

In fact, $\mathfrak{w}$ is defined so that the spectral curve (4.1) can be reconstructed from the characteristic equation of $\mathbb{L}$:

$$0=\det\big[\mathbb{L}-\lambda^m\mu\mathbb{I}\big]=-(\mathfrak{v}^2+\mathfrak{u}\mathfrak{w})+\lambda^{2m}\mu^2=-\lambda^m\left(\sigma(\lambda)+\sum_{k=1}^{n}H_k\lambda^{n-k}\right)+\lambda^{2m}\mu^2. \tag{4.15}$$

One can show that the expression in the quadratic bracket in (4.14) factorizes so that $\mathfrak{w}$ takes the form of a Laurent polynomial in λ:

$$\mathfrak{w}=\lambda^m\left[\frac{\sigma(\lambda)-\lambda^{-m}\mathfrak{v}^2}{\mathfrak{u}}\right]_+. \tag{4.16}$$

Here, the operation $[\cdot]_+$ is defined as follows. Given an analytic function ψ and a (pure) polynomial $\mathfrak{u}$,

[3] In general the Lax matrices for Benenti systems are parametrized by two arbitrary functions $f(\lambda)$ and $g(\lambda)$, see [7]. In this paper we choose these functions as $g=\frac{1}{2}f=\lambda^m$.

we define $\left[\frac{\psi}{\mathfrak{u}}\right]_+$ so that the following decomposition holds:

$$\psi = \left[\frac{\psi}{\mathfrak{u}}\right]_+ \mathfrak{u} + r,$$

where the (unique) remainder r is a lower degree polynomial than the polynomial $\mathfrak{u}$, see [7] for details. In particular, when ψ is a Laurent polynomial, we have

$$\left[\frac{\psi}{\mathfrak{u}}\right]_+ \equiv \left[\frac{[\psi]_{\geqslant 0}}{\mathfrak{u}}\right]_{\geqslant 0} + \left[\frac{[\psi]_{<0}}{\mathfrak{u}}\right]_{<0}, \tag{4.17}$$

where $[\cdot]_{\geqslant 0}$ is the projection on the part consisting of non-negative degree terms in the expansion into its Laurent series at ∞ and $[\cdot]_{<0}$ is the projection on the part consisting of negative degree terms in the expansion into its Laurent series at 0.

The auxiliary matrices $\mathbb{U}_k$ are defined by

$$\mathbb{U}_k := \left[\frac{\mathfrak{u}_k \mathbb{L}}{\mathfrak{u}}\right]_+ \equiv \begin{pmatrix} \left[\frac{\mathfrak{u}_k \mathfrak{v}}{\mathfrak{u}}\right]_+ & \mathfrak{u}_k \\ \left[\frac{\mathfrak{u}_k \mathfrak{w}}{\mathfrak{u}}\right]_+ & -\left[\frac{\mathfrak{u}_k \mathfrak{v}}{\mathfrak{u}}\right]_+ \end{pmatrix}, \qquad k = 1, \ldots, n, \tag{4.18}$$

where

$$\mathfrak{u}_k := \left[\frac{\mathfrak{u}}{\lambda^{n-k+1}}\right]_+ \equiv \lambda^{k-1} + \sum_{i=1}^{k-1} (-1)^k s_k \lambda^{k-i-1}.$$

The Lax equations (4.12) (describing the evolution of the Lax matrix (4.13) with respect to Hamiltonian equations (4.6)), can be derived from the evolution equations for $\mathfrak{u}$,$\mathfrak{v}$,$\mathfrak{w}$:

$$\mathfrak{u}_{t_k} \equiv \{\mathfrak{u}, H_k\} = -2\mathfrak{u}_k \mathfrak{v} + 2\mathfrak{u} \left[\frac{\mathfrak{u}_k \mathfrak{v}}{\mathfrak{u}}\right]_+, \tag{4.19a}$$

$$\mathfrak{v}_{t_k} \equiv \{\mathfrak{v}, H_k\} = \mathfrak{u}_k \mathfrak{w} - \mathfrak{u} \left[\frac{\mathfrak{u}_k \mathfrak{w}}{\mathfrak{u}}\right]_+, \tag{4.19b}$$

$$\mathfrak{w}_{t_k} \equiv \{\mathfrak{w}, H_k\} = -2\mathfrak{w} \left[\frac{\mathfrak{u}_k \mathfrak{v}}{\mathfrak{u}}\right]_+ + 2\mathfrak{v} \left[\frac{\mathfrak{u}_k \mathfrak{w}}{\mathfrak{u}}\right]_+, \tag{4.19c}$$

which were obtained in [7].

Since $\mathfrak{u}_1 = 1$ and $\left[\frac{\mathfrak{v}}{\mathfrak{u}}\right]_+ = 0$, the equations (4.19a) and (4.19b) for $k = 1$ read

$$\dot{\mathfrak{u}} = -2\mathfrak{v}, \qquad \dot{\mathfrak{v}} = \mathfrak{w} + \mathfrak{u}\mathcal{Q}, \qquad \mathcal{Q} := -\left[\frac{\mathfrak{w}}{\mathfrak{u}}\right]_+.$$

Here and in what follows, the dot means the derivative with respect to the first Hamiltonian flow, i.e. $\dot{\xi} \equiv \xi_{t_1}$. Hence, we can rewrite the Lax matrix (4.13) as

$$\mathbb{L} = \begin{pmatrix} -\frac{1}{2}\dot{\mathfrak{u}} & \mathfrak{u} \\ -\frac{1}{2}\ddot{\mathfrak{u}} - \mathfrak{u}\mathcal{Q} & \frac{1}{2}\dot{\mathfrak{u}} \end{pmatrix}, \tag{4.20}$$

(see also [18]). Further, observing that

$$\dot{\mathfrak{u}}_k = \left[\frac{\mathfrak{u}_k \dot{\mathfrak{u}}}{\mathfrak{u}}\right]_+ = -2\left[\frac{\mathfrak{u}_k \mathfrak{v}}{\mathfrak{u}}\right]_+$$

and

$$\ddot{\mathfrak{u}}_k = \left[\frac{\mathfrak{u}_k \ddot{\mathfrak{u}}}{\mathfrak{u}}\right]_+ = -2\left[\frac{\mathfrak{u}_k \dot{\mathfrak{v}}}{\mathfrak{u}}\right]_+ = -2\left[\frac{\mathfrak{u}_k \mathfrak{w}}{\mathfrak{u}}\right]_+ - 2\left[\frac{\mathfrak{u}_k \mathfrak{u}\mathcal{Q}}{\mathfrak{u}}\right]_+ = -2\left[\frac{\mathfrak{u}_k \mathfrak{w}}{\mathfrak{u}}\right]_+ - 2\mathfrak{u}_k\mathcal{Q},$$

we can rewrite the auxiliary matrices (4.18) in the form

$$\mathbb{U}_k = \begin{pmatrix} -\frac{1}{2}\dot{\mathfrak{u}}_k & \mathfrak{u}_k \\ -\frac{1}{2}\ddot{\mathfrak{u}}_k - \mathfrak{u}_k\mathcal{Q} & \frac{1}{2}\dot{\mathfrak{u}}_k \end{pmatrix}, \qquad k = 1, \ldots, n. \tag{4.21}$$

(note that also in this notation $\mathbb{U}_1 = \mathbb{L}$, as it should be). The characteristic equation (4.15) for $\mathbb{L}$ in the form (4.20) is given by

$$-\frac{1}{2}\mathfrak{u}\ddot{\mathfrak{u}} + \frac{1}{4}\dot{\mathfrak{u}}^2 - \mathfrak{u}^2\mathcal{Q} = \lambda^{2m}\mu^2. \tag{4.22}$$

Using the form (4.20) of $\mathbb{L}$ and the form (4.21) of $\mathbb{U}_k$ we see that the Lax equation (4.12) , for $k = 1$ yields the following ODE:

$$\dddot{\mathfrak{u}} + 4\dot{\mathfrak{u}}\mathcal{Q} + 2\mathfrak{u}\dot{\mathcal{Q}} = 0, \tag{4.23}$$

while the remaining (i.e. for $k = 2, \ldots, n$) Lax equations in (4.12) yield, due to (4.23), the following hierarchy of PDE's:

$$\mathfrak{u}_{t_k} = \dot{\mathfrak{u}}\mathfrak{u}_k - \mathfrak{u}\dot{\mathfrak{u}}_k, \tag{4.24}$$

$$\mathcal{Q}_{t_k} = \frac{1}{2}\dddot{\mathfrak{u}}_k + 2\dot{\mathfrak{u}}_k\mathcal{Q} + \mathfrak{u}_k\dot{\mathcal{Q}}. \tag{4.25}$$

Notice that the equation (4.23) can also be obtained by differentiating (4.22) with respect to time t_1. Besides, using (4.19a), equations (4.24) are identically satisfied.

Now, comparing (4.20–4.21) and (2.32) opens the possibility of constructing a map between Stäckel systems defined by the curve (3.31) and an appropriate stationary cKdV system. This map will be constructed in the next section.

The functions $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ can also be written in Viète's coordinates (4.7). The function $\mathfrak{u}$ is given by

$$\mathfrak{u} = \lambda^n + \sum_{k=1}^{n} q_k\lambda^{n-k},$$

$$\mathfrak{v} = -\frac{1}{2}\sum_{k=1}^{n}\Big[\sum_{l=1}^{n}(G_m)^{kl}p_l\Big]\lambda^{n-k} \equiv -\frac{1}{2}\dot{\mathfrak{u}}.$$

while the function $\mathfrak{w}$ can still be obtained from the formula (4.14) or (4.16).

5 Equivalent Stäckel representations of stationary cKdV systems on $\mathcal{M}_n$

We are finally in position to formulate and prove the main result of this paper: the two seemingly distinct objects, stationary cKdV systems and Stäckel systems of Benenti type, are in fact the same, i.e. represent the same finite dimensional integrable system written in two different systems of coordinates. More precisely, we will prove below that each stationary N-field cKdV system on $\mathcal{M}_n$ can be identified, on $N+1$ equivalent ways, with an appropriate Stäckel system of Benenti type.

Let us thus fix $m \in \{0, 1, \ldots, N\}$ and consider the separation curve (3.31),

$$\lambda^{2n+N-m} + \sum_{k=1}^{N-m} c_{m+k}\lambda^{n+k-1} + \sum_{k=1}^{n} H_k\lambda^{n-k} + \sum_{k=1}^{m} c_k\lambda^{k-m-1} = \lambda^m\mu^2, \tag{5.1}$$

associated with m-th foliation (3.8) of the n-th stationary cKdV system (3.3). Comparing (5.1) with (4.1) we see that

$$\sigma(\lambda) = \lambda^{2n+N-m} + \sum_{k=1}^{N-m} c_{m+k}\lambda^{n+k-1} + \sum_{k=1}^{m} c_k\lambda^{k-m-1}.$$

The identification we want to achieve will be done through an appropriate map between the jet variables $[\boldsymbol{u}]$ of the stationary cKdV system and coordinates of the Stäckel system (5.1) defined on the (extended) Poisson manifold of dimension $2n+N$, spanned by the coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c})$ and equipped with the Poisson bracket $\pi = \sum_i \frac{\partial}{\partial\lambda_i} \wedge \frac{\partial}{\partial\mu_i}$ (so that $\boldsymbol{c} = (c_1, \ldots, c_N)$ are Casimir variables for π). Thus, this identification will depend on the choice of $m \in \{0, 1, \ldots, N\}$ and can therefore be obtained on $N+1$ different ways.

Let us now explicitly write the Stäckel system defined by the curve (5.1). Solving (5.1) with respect to H_k and passing to (extended) Viète coordinates $(\boldsymbol{q},\boldsymbol{p},\boldsymbol{c})$ we obtain n Stäckel Hamiltonians $H_k = H_k(\boldsymbol{q},\boldsymbol{p},\boldsymbol{c})$, $k=1,\ldots,n$, of the form (4.8) with potentials V_k given by

$$V_k(\boldsymbol{q},\boldsymbol{c}) = \mathcal{V}_k^{(2n+N-m)} + \sum_{i=1}^{N-m} c_{m+i}\mathcal{V}_k^{(n+i-1)} + \sum_{i=1}^{m} c_k\mathcal{V}_k^{(i-m-1)}.$$

The Stäckel Hamiltonians H_k on the extended phase space generate the following Stäckel system

$$\boldsymbol{q}_{t_k} = \{\boldsymbol{q},H_k\} \equiv \frac{\partial H_k}{\partial \boldsymbol{p}}, \qquad \boldsymbol{p}_{t_k} = \{\boldsymbol{p},H_k\} \equiv -\frac{\partial H_k}{\partial \boldsymbol{q}}, \qquad \boldsymbol{c}_{t_k} = \{\boldsymbol{c},H_k\} \equiv 0, \qquad k=1,\ldots,n. \tag{5.2}$$

In what follows we will need the following lemma.

Lemma 5.1. *The element $\mathcal{Q}$ from the Lax matrices* (4.20) *and* (4.21) *associated with the separation curve* (5.1) *is a polynomial of order N:*

$$\mathcal{Q} \equiv -\left[\frac{\mathfrak{w}}{\mathfrak{u}}\right]_+ = -\left[\frac{\mathfrak{w}}{\mathfrak{u}}\right]_{\geqslant 0} = -\lambda^N + l.d.t., \tag{5.3}$$

where l.d.t. denotes the lower degree terms with coefficients being functions of $(\boldsymbol{q},\boldsymbol{p},\boldsymbol{c})$.

Proof. The relation (5.3) is a direct consequence of the identity (4.17) and the fact that $\mathfrak{w}$ is a polynomial. □

We are now in position to formulate and prove the main theorem of this paper.

Theorem 5.1. *For a given $m \in \{0,1,\ldots,N\}$, the transformation between the jet variables $[\mathbf{u}]$ on the stationary manifold $\mathcal{M}_n$ and the Viète coordinates $(\boldsymbol{q},\boldsymbol{p},\boldsymbol{c})$ given by*

$$q_i = \frac{1}{2}P_i, \qquad p_i = \frac{1}{2}\sum_{j=1}^{n}\left(G_m^{-1}\right)_{ij}(P_j)_x, \qquad i=1,\ldots,n, \tag{5.4a}$$

and

$$c_i = h_{i-1}, \qquad i=1,\ldots,m, \qquad c_i = h_{n+i-1}, \qquad i=m+1,...,N, \tag{5.4b}$$

maps (after the identification $t_1 \equiv x$) the r-th flow $\boldsymbol{u}_{t_r} = \boldsymbol{K}_r$ of the stationary cKdV system (3.3) on $\mathcal{M}_n$ onto the r-th flow of the Stäckel system (5.2). The metric G_m in (5.4) is expressed in coordinates $q_i = q_i[\mathbf{u}]$ that are given by the first formula in (5.4a).

Proof. Let us make the following identification

$$\mathfrak{u} \equiv \frac{1}{2}\mathbb{P}_{n+1}$$

between the variables q_i and the jet variables $[\boldsymbol{u}]$. This formula immediately yields the first part of the map (5.4). Extending it to the momenta part we immediately obtain the second part (this part depends on m) of (5.4). By Lemma 5.1 also $\mathcal{Q}$ and $\mathbb{Q}$ must then coincide (the explicit identification between $\mathcal{Q}$ and $\mathbb{Q}$ also depends on m). Thus further, after the identification $t_1 = x$ and by comparing $\mathbb{L}$ in (4.20) with $\mathbb{V}_{n+1}$ in (2.32) and $\mathbb{U}_k$ in (4.21) with respective $\mathbb{V}_k$ in (2.32) we see that on the particular leaf $\mathcal{M}^{\boldsymbol{c}}_{n,m}$ we must have

$$\mathbb{L} \equiv \mathbb{V}_{n+1}^{(m)}, \qquad \mathbb{U}_k \equiv \mathbb{V}_k^{(m)}, \qquad k=1,\ldots,n,$$

so that Lax formulations of both systems coincide. Moreover, (4.22) coincides with (3.27), (4.23) coincides with (3.9) and (4.25) coincides with (2.2). Finally, the map (5.4b) is given by (3.16). Thus, the whole map (5.4) maps the Stäckel system defined by the curve (5.1) to the corresponding stationary cKdV system (3.3) on $\mathcal{M}_n$. □

Remark 5.1. Let us remark that the map (5.4a) is defined on each leaf $\mathcal{M}^{\boldsymbol{c}}_{n,m}$ while the whole map (5.4) is defined on the stationary manifold $\mathcal{M}_n$. Thus, (5.4a) maps the r-th flow $\boldsymbol{u}_{t_r} = \boldsymbol{K}_r$ on the leaf $\mathcal{M}^{\boldsymbol{c}}_{n,m}$ onto the r-th flow of the Stäckel system (4.9) (we remind the reader that each cKdV flow $\boldsymbol{u}_{t_r} = \boldsymbol{K}_r$ is tangent not only to $\mathcal{M}_n$ but also to each leaf $\mathcal{M}^{\boldsymbol{c}}_{n,m}$ of the respective foliation).

6 Miura maps between Stäckel representations on $\mathcal{M}_n$

In this chapter we prove that all $N+1$ Stäckel representations (5.2) of the stationary cKdV system (3.3), when considered on the whole stationary manifold $\mathcal{M}_n$ and not only on the leaves $M^{\boldsymbol{c}}_{n,m}$ (that is on the extended phase space when $\boldsymbol{c}$ are variables on $\mathcal{M}_n$ rather than parameters of the foliation $M^{\boldsymbol{c}}_{n,m}$) are connected by appropriate Miura maps. To simplify the presentation, we will only consider the case when one of the representations is given by $m=0$ (the general case does not create any problems whatsoever). Consider thus two Stäckel representations of the same cKdV stationary system (3.3) on $\mathcal{M}_n$ generated by the curves

$$\lambda^{2n+N}+\sum_{k=1}^{N}c_k\lambda^{n+k-1}+\sum_{k=1}^{n}H_k\lambda^{n-k}=\mu^2 \tag{6.1}$$

and

$$\bar{\lambda}^{2n+N-m}+\sum_{k=1}^{N-m}\bar{c}_{m+k}\bar{\lambda}^{n+k-1}+\sum_{k=1}^{n}\bar{H}_k\bar{\lambda}^{n-k}+\sum_{k=1}^{m}\bar{c}_k\bar{\lambda}^{k-m-1}=\bar{\lambda}^m\bar{\mu}^2, \tag{6.2}$$

with $m\in\{1,\ldots,N\}$. We will thus consider the Stäckel system generated by the curve (6.1) (it has the form (5.2)) in the extended Viète coordinates $(\boldsymbol{q},\boldsymbol{p},\boldsymbol{c})$, with $(\boldsymbol{q},\boldsymbol{p})$ still given by (4.7) and we will also consider the Stäckel system generated by the curve (6.2) (it also has the form (5.2)) in the extended Viète coordinates $(\bar{\boldsymbol{q}},\bar{\boldsymbol{p}},\bar{\boldsymbol{c}})$ where $(\bar{\boldsymbol{q}},\bar{\boldsymbol{p}})$ are Viète coordinates defined by $\bar{\lambda}_k$ and $\bar{\mu}_k$. The theorem below shows that these systems are indeed two different parametrizations of the same system, connected by a Miura map, as it should be, since both are connected by two invertible maps (5.4) with the same stationary cKdV system (2.9).

Theorem 6.1. *The following Miura map on the stationary manifold $\mathcal{M}_n$*

$$\begin{aligned}
&\boldsymbol{q}=\bar{\boldsymbol{q}}\\
&\boldsymbol{p}=\left(R^T\right)^m\bar{\boldsymbol{p}},\qquad \left[\left(R^T\right)^m\right]_{ij}=\mathcal{V}_j^{(n-i+m)}(\bar{\boldsymbol{q}}),\quad i,j=1,...,n,\\
&c_i=\bar{H}_{m-i+1}(\bar{\boldsymbol{q}},\bar{\boldsymbol{p}},\bar{\boldsymbol{c}}),\qquad i=1,\ldots,m,\\
&c_i=\bar{c}_i,\quad i=m+1,\ldots,N
\end{aligned} \tag{6.3}$$

transforms the Stäckel system defined by the curve (6.2) to the Stäckel system generated by the curve (6.1). Similarly, the inverse of this map, given by

$$\begin{aligned}
&\bar{\boldsymbol{q}}=\boldsymbol{q}\\
&\bar{\boldsymbol{p}}=\left(R^T\right)^{-m}\boldsymbol{p},\qquad \left[\left(R^T\right)^m\right]_{ij}=\mathcal{V}_j^{(n-i+m)}(\boldsymbol{q}),\quad i,j=1,...,n,\\
&\bar{c}_i=H_{n-i+1}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{c}),\qquad i=1,\ldots,m\\
&\bar{c}_i=c_i,\qquad i=m+1,\ldots,N.
\end{aligned}$$

transforms the Stäckel system defined by the curve (6.1) to the Stäckel system generated by the curve (6.2).

Proof. For the fixed $m\in\{1,\ldots,N\}$, the map

$$\lambda=\bar{\lambda},\qquad \mu=\bar{\lambda}^m\bar{\mu} \tag{6.4}$$

transforms the curve (6.2) into the curve (6.1), provided that

$$\begin{aligned}
c_i&=\bar{c}_i, && i=m+1,\ldots,N\\
c_i&=\bar{H}_{m-i+1}, && i=1,\ldots,m\\
H_i&=\bar{H}_{m+i}, && i=1,\ldots,n-m\\
H_i&=\bar{c}_{n-i+1}, && i=n-m+1,\ldots,n.
\end{aligned} \tag{6.5}$$

As result, the maps (6.4) and (6.5) induce the following Miura map $\mathfrak{M}: \mathcal{M}_n \to \mathcal{M}_n$

$$\begin{aligned} \lambda_i &= \bar{\lambda}_i, \qquad i=1,\dots,n \\ \mu_i &= \bar{\lambda}_i^m \bar{\mu}_i, \qquad i=1,\dots,n \\ c_i &= \bar{H}_{m-i+1}(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{c}}), \qquad i=1,\dots,m \\ c_i &= \bar{c}_i, \qquad i=m+1,\dots. \end{aligned} \tag{6.6}$$

Let us now write the Miura map (6.6) in Viète coordinates. Since $\lambda_i = \bar{\lambda}_i$ we have $q_i = \bar{q}_i$. Further, since $\mu_i = \bar{\lambda}_i^m \bar{\mu}_i$, we obtain

$$p_i = -\sum_{k=1}^n \frac{\lambda_k^{n-i}\mu_k}{\Delta_k} = -\sum_{k=1}^n \frac{\bar{\lambda}_k^{n-i+m}\bar{\mu}_k}{\bar{\Delta}_k} = \sum_{k=1}^n \mathcal{V}_k^{(n-i+m)}(\bar{\boldsymbol{q}})\bar{p}_k,$$

where the last equality follows from (4.11). Alternatively, we see that the first Hamiltonian flows of the corresponding Stäckel systems are $\dot{\boldsymbol{q}} = G_0\boldsymbol{p}$ and $\dot{\bar{\boldsymbol{q}}} = G_m\bar{\boldsymbol{p}}$.[4] Now, since $\dot{\boldsymbol{q}} = \dot{\bar{\boldsymbol{q}}}$ and $G_m = R^m G_0 = G_0 \left(R^T\right)^m$ we have

$$\boldsymbol{p} = G_0^{-1} G_m \bar{\boldsymbol{p}} = \left(R^T\right)^m \bar{\boldsymbol{p}}.$$

Rewriting the Hamiltonians $\bar{H}_m$ in the coordinates $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{c}})$ completes expressing the Miura map (6.6) in Viéte coordinates.

The proof of the second statement of the theorem is analogous. The relations (6.5) can be inverted to

$$\begin{aligned} \bar{c}_i &= H_{n-i+1}, \qquad i=1,\dots,m \\ \bar{c}_i &= c_i, \qquad i=m+1,\dots,N \\ \bar{H}_i &= c_{m-i+1}, \qquad i=1,\dots,m \\ \bar{H}_i &= H_{i-m}, \qquad i=m+1,\dots,n. \end{aligned} \tag{6.7}$$

The maps (6.4) and (6.7) induce the inverse Miura map $\mathfrak{M}^{-1}: \mathcal{M}_n \to \mathcal{M}_n$ given by

$$\begin{aligned} \bar{\lambda}_i &= \lambda_i, \qquad i=1,\dots,n \\ \bar{\mu}_i &= \lambda_i^{-m}\mu_i, \qquad i=1,\dots,n \\ \bar{c}_i &= H_{n-i+1}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}), \qquad i=1,\dots,m \\ \bar{c}_i &= c_i, \qquad i=m+1,\dots,N, \end{aligned} \tag{6.8}$$

The Miura map $\mathfrak{M}$ maps the Stäckel system generated by the curve (6.2) into the Stäckel system generated by the curve (6.1), while the map $\mathfrak{M}^{-1}$ maps this system back into the system generated by (6.2). □

Obviously, $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{c})$ are canonical coordinates with respect to the Poisson bi-vector $\pi = \sum_{i=1}^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$ while the coordinates $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{c}})$ are canonical coordinates with respect to the Poisson bi-vector $\bar{\pi} = \sum_{i=1}^n \frac{\partial}{\partial \bar{q}_i} \wedge \frac{\partial}{\partial \bar{p}_i}$. Since there exists a Miura map between (6.1) and (6.2) for each $m \in \{1,\dots,N\}$, we can construct, in a standard way, $N+1$ Poisson bi-vectors for the Stäckel system (6.1) on $\mathcal{M}_n$, each given by rewriting the bi-vector $\bar{\pi}$ in the coordinates $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{c})$, yielding $N+1$ Hamiltonian representations of the Stäckel system (6.1). Thus, we have proved that the $N+1$ Hamiltonian representations of cKdV hierarchy leads to $N+1$ Hamiltonian representations of its stationary system. Let us note that the very existence of Miura maps between stationary systems (and hence the multi-Hamiltonian structure of these systems) has been first observed in [2] for the KdV case.

[4] Note that we are not distinguishing between metrics defined in coordinates q_i and $\bar{q}_i$ because $q_i \equiv \bar{q}_i$, analogously in the case of the matrix R.

7 Examples

In this final section we present examples illustrating all main ingredients of the theory. Subsection 7.1 focuses on the DWW hierarchy (so that $N = 2$) and contains two examples. The first example is for $n = 2$ and all $m = 0, 1, 2$ and is given with all details. In the second example of this subsection we present the case $n = 3$ and $m = 0$. In subsection 7.2 we illustrate our theory for the case $N = 4$, $n = 2$ and $m = 0$. This example aims to show what happens in case we reduce the cKdV hierarchy to stationary manifolds $\mathcal{M}_n$ of low (in comparison to the number N of components) dimension.

7.1 Dispersive Water Waves hierarchy

Assume that $N = 2$ and denote $\boldsymbol{u} := (u_0, u_1)^T = (u, v)^T$. Then the formulas from Section 2 lead to the DWW hierarchy, the first members of which are given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \boldsymbol{K}_1 \equiv \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \boldsymbol{K}_2 \equiv \begin{pmatrix} \frac{1}{2}vu_x + uv_x + \frac{1}{4}v_{3x} \\ u_x + \frac{3}{2}vv_x \end{pmatrix}, \tag{7.1a}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_3} = \boldsymbol{K}_3 \equiv \begin{pmatrix} \frac{3}{8}v^2u_x + \frac{3}{2}uvv_x + \frac{3}{2}uu_x + \frac{1}{4}u_{3x} + \frac{3}{8}vv_{3x} + \frac{9}{8}v_xv_{2x} \\ \frac{3}{2}vu_x + \frac{3}{2}uv_x + \frac{15}{8}v^2v_x + \frac{1}{4}v_{3x} \end{pmatrix}, \tag{7.1b}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_4} = \boldsymbol{K}_4 \equiv \begin{pmatrix} (\boldsymbol{K}_4)_1 \\ \frac{15}{8}v^2u_x + \frac{15}{4}uvv_x + \frac{3}{2}uu_x + \frac{1}{4}u_{3x} + \frac{35}{16}v^3v_x + \frac{5}{8}vv_{3x} + \frac{5}{4}v_xv_{2x} \end{pmatrix}, \tag{7.1c}$$

where

$$\begin{aligned}(\boldsymbol{K}_4)_1 = {} & \frac{3}{2}u^2v_x + \frac{5}{16}v^3u_x + \frac{15}{8}uv^2v_x + \frac{9}{4}uvu_x + \frac{3}{8}vu_{3x} + \frac{9}{8}u_{2x}v_x + \frac{5}{4}u_xv_{2x} + \frac{5}{8}uv_{3x} \\ & + \frac{15}{32}v^2v_{3x} + \frac{45}{16}vv_xv_{2x} + \frac{15}{16}v_x^3 + \frac{1}{16}v_{5x}\end{aligned}$$

This hierarchy, according to (2.20), is three-hamiltonian with the Hamiltonian operators given by

$$\mathbb{B}_0 = \begin{pmatrix} -\frac{1}{2}v\partial_x - \frac{1}{2}\partial_x v & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad \mathbb{B}_1 = \begin{pmatrix} \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & 0 \\ 0 & \partial_x \end{pmatrix},$$

$$\mathbb{B}_2 = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u \\ \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & \frac{1}{2}v\partial_x + \frac{1}{2}\partial_x v \end{pmatrix}.$$

The cosymmetries $\boldsymbol{\gamma}_k = (P_k, P_{k+1})^T$ are given by

$$\begin{aligned}
& P_0 = 2, \qquad P_1 = v, \qquad P_2 = u + \frac{3}{4}v^2, \qquad P_3 = \frac{3}{2}uv + \frac{5}{8}v^3 + \frac{1}{4}v_{2x}, \\
& P_4 = \frac{3}{4}u^2 + \frac{15}{8}uv^2 + \frac{1}{4}u_{2x} + \frac{35}{64}v^4 + \frac{5}{8}vv_{2x} + \frac{5}{16}v_x^2, \\
& P_5 = \frac{15}{8}u^2v + \frac{35}{16}uv^3 + \frac{5}{8}vu_{2x} + \frac{5}{8}u_xv_x + \frac{5}{8}uv_{2x} + \frac{63}{128}v^5 + \frac{35}{32}v^2v_{2x} + \frac{35}{32}vv_x^2 + \frac{1}{16}v_{4x}, \\
& \quad \vdots
\end{aligned}$$

while the recursion operator (2.21) attains the form

$$\mathbb{R} = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1} \\ 1 & v + \frac{1}{2}v_x\partial_x^{-1} \end{pmatrix}.$$

Moreover, the Lax matrix $\mathbb{V}_1$ and the first three auxiliary matrices $\mathbb{V}_k$ are as follows

$$\mathbb{V}_1 = \begin{pmatrix} 0 & 1 \\ \lambda^2 - v\lambda - u & 0 \end{pmatrix}, \qquad \mathbb{V}_2 = \begin{pmatrix} -\frac{1}{4}v_x & \lambda + \frac{1}{2}v \\ \lambda^3 - \frac{1}{2}v\lambda^2 - \left(u + \frac{1}{2}v^2\right)\lambda - \frac{1}{2}uv - \frac{1}{4}v_{2x} & \frac{1}{4}v_x \end{pmatrix},$$

$$\mathbb{V}_3 = \begin{pmatrix} -\frac{1}{4}v_x\lambda - \frac{1}{4}u_x - \frac{3}{8}vv_x & \lambda^2 + \frac{1}{2}v\lambda + \frac{3}{8}v^2 + \frac{1}{2}u \\ (\mathbb{V}_3)_{21} & \frac{1}{4}v_x\lambda + \frac{1}{4}u_x + \frac{3}{8}vv_x \end{pmatrix},$$

where

$$(\mathbb{V}_3)_{21} = \lambda^4 - \frac{1}{2}v\lambda^3 - \frac{1}{8}\left(4u + v^2\right)\lambda^2 - \frac{1}{8}\left(8uv + 3v^3 + 2v_{2x}\right)\lambda - \frac{1}{8}\left(4u^2 + 3uv^2 + 2u_{2x} + 3v_x^2 + 3vv_{2x}\right),$$

and

$$\mathbb{V}_4 = \begin{pmatrix} (\mathbb{V}_4)_{11} & \lambda^3 + \frac{1}{2}v\lambda^2 + \frac{1}{8}\left(3v^2 + 4u\right)\lambda + \frac{1}{16}\left(5v^3 + 12uv + 2v_{2x}\right) \\ (\mathbb{V}_4)_{21} & -(\mathbb{V}_4)_{11} \end{pmatrix},$$

where

$$\begin{aligned}
(\mathbb{V}_4)_{11} &= -\frac{1}{4}v_x\lambda^2 - \frac{1}{8}\left(2u_x + 3vv_x\right)\lambda - \frac{1}{32}\left(12vu_x + 12uv_x + 15v^2v_x + 2v_{3x}\right),\\
(\mathbb{V}_4)_{21} &= \lambda^5 - \frac{1}{2}v\lambda^4 - \frac{1}{8}\left(4u + v^2\right)\lambda^3 - \frac{1}{16}\left(4uv + v^3 + 2v_{2x}\right)\lambda^2\\
&\quad - \frac{1}{16}\left(8u^2 + 18uv^2 + 4u_{2x} + 5v^4 + 8vv_{2x} + 6v_x^2\right)\lambda\\
&\quad - \frac{1}{16}\left(12u^2v + 5uv^3 + 6vu_{2x} + 12u_xv_x + 8uv_{2x} + \frac{15}{2}v^2v_{2x} + 15vv_x^2 + v_{4x}\right).
\end{aligned}$$

7.1.1 The stationary reduction with $n = 2$ and $m = 0, 1, 2$

Here we consider three Stäckel representations of the stationary DWW system for $n = 2$.

Stationary system

For $N = 2$ and $n = 2$ the stationary system (3.3) consists of two evolution equations (7.1a)

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \boldsymbol{K}_1 \equiv \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \boldsymbol{K}_2 \equiv \begin{pmatrix} \frac{1}{2}vu_x + uv_x + \frac{1}{4}v_{3x} \\ u_x + \frac{3}{2}vv_x \end{pmatrix} \tag{7.2}$$

and of the stationary flow $\boldsymbol{K}_3 = 0$,

$$\begin{pmatrix} \frac{3}{8}v^2u_x + \frac{3}{2}uvv_x + \frac{3}{2}uu_x + \frac{1}{4}u_{3x} + \frac{3}{8}vv_{3x} + \frac{9}{8}v_xv_{2x} \\ \frac{3}{2}vu_x + \frac{3}{2}uv_x + \frac{15}{8}v^2v_x + \frac{1}{4}v_{3x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The normal form of the stationary flow $\boldsymbol{K}_3 = 0$ is

$$\begin{aligned}
u_{3x} &= -6uu_x + \frac{15}{2}v^2u_x + 3uvv_x + \frac{45}{4}v^3v_x - \frac{9}{2}v_xv_{2x}\\
v_{3x} &= -6vu_x - 6uv_x - \frac{15}{2}v^2v_x
\end{aligned}$$

and yields us the stationary manifold $\mathcal{M}_2$ parameterized by the jet variables $[\boldsymbol{u}] = (u, u_x, u_{2x}, v, v_x, v_{2x})$. The vector fields (7.2) in this parametrization attain on $\mathcal{M}_2$ the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} -vu_x - \frac{1}{2}uv_x - \frac{15}{8}v^2v_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix}. \tag{7.3}$$

The corresponding separation curve (3.29) attains the form

$$\lambda^6 + h_3\lambda^3 + h_2\lambda^2 + h_1\lambda + h_0 = \lambda^{2m}\mu^2,$$

which yields the following h_k in (3.13) on $\mathcal{M}_2$

$$
\begin{aligned}
h_3 &= -\frac{3}{2}uv - \frac{1}{4}v_{2x} - \frac{5}{8}v^3,\\
h_2 &= -\frac{9}{8}uv^2 - \frac{1}{4}u_{2x} - \frac{3}{4}u^2 - \frac{1}{2}vv_{2x} - \frac{5}{16}v_x^2 - \frac{15}{64}v,\\
h_1 &= -\frac{1}{8}vu_{2x} + \frac{1}{8}u_xv_x - \frac{1}{8}uv_{2x} - \frac{3}{4}uv^3 - \frac{3}{4}u^2v - \frac{9}{32}v^2v_{2x} - \frac{9}{64}v^5,\\
h_0 &= -\frac{3}{32}v^2u_{2x} + \frac{3}{16}vu_xv_x - \frac{3}{16}uvv_{2x} - \frac{3}{16}uv_x^2 - \frac{9}{64}uv^4 - \frac{3}{8}u^2v^2 + \frac{1}{16}u_x^2 - \frac{1}{8}uu_{2x} - \frac{1}{4}u^3 - \frac{9}{64}v^3v_{2x}.
\end{aligned}
$$

Foliation for $m = 0$

Consider first the case $m = 0$. Putting $h_3 = c_2$ and $h_2 = c_1$ we obtain the foliation of $\mathcal{M}_2$ into leaves $\mathcal{M}_{2,0}^{\mathbf{c}}$. Solving these relations with respect to u_{2x} and v_{2x},

$$
u_{2x} = 8c_2v - 4c_1 - 3u^2 + \frac{15}{2}uv^2 + \frac{65}{16}v^4 - \frac{5}{4}v_x^2, \qquad v_{2x} = -4c_2 - 6uv - \frac{5}{2}v^3,
$$

we arrive at the curve (3.31) for the leaves $\mathcal{M}_{2,0}^{\mathbf{c}}$. It has the form

$$
\lambda^6 + c_2\lambda^3 + c_1\lambda^2 + H_1\lambda + H_2 = \mu^2 \tag{7.4}
$$

with H_i given by

$$
\begin{aligned}
H_1 &= \frac{3}{8}u^2v + \frac{5}{16}uv^3 + \frac{1}{8}u_xv_x + \frac{7}{128}v^5 + \frac{5}{32}vv_x^2 + \frac{1}{2}c_2u + \frac{1}{8}c_2v^2 + \frac{1}{2}c_1v,\\
H_2 &= +\frac{1}{8}u^3 + \frac{3}{32}u^2v^2 - \frac{5}{128}uv^4 + \frac{3}{16}vu_xv_x - \frac{1}{32}uv_x^2 + \frac{1}{16}u_x^2 - \frac{15}{512}v^6\\
&\quad + \frac{15}{128}v^2v_x^2 - \frac{1}{4}c_2uv + \frac{1}{2}c_1u - \frac{3}{16}c_2v^3 + \frac{3}{8}c_1v^2.
\end{aligned}
$$

The Lax matrices $\mathbb{V}_k^{(0)}$ in (3.25) are as follows, $\mathbb{V}_1^{(0)} = \mathbb{V}_1$,

$$
\mathbb{V}_2^{(0)} = \begin{pmatrix} -\frac{1}{4}v_x & \frac{1}{2}v + \lambda \\ \lambda^3 - \frac{1}{2}v\lambda^2 - \left(u + \frac{1}{2}v^2\right)\lambda + uv + \frac{5}{8}v^3 + c_2 & \frac{1}{4}v_x \end{pmatrix}
$$

$$
\mathbb{V}_3^{(0)} = \begin{pmatrix} -\frac{1}{4}v_x\lambda - \frac{1}{4}u_x - \frac{3}{8}vv_x & \lambda^2 + \frac{1}{2}v\lambda + \frac{3}{8}v^2 + \frac{1}{2}u \\ \left(\mathbb{V}_3^{(0)}\right)_{21} & \frac{1}{4}v_x\lambda + \frac{1}{4}u_x + \frac{3}{8}vv_x \end{pmatrix},
$$

where

$$
\left(\mathbb{V}_3^{(0)}\right)_{21} = \lambda^4 - \frac{1}{2}v\lambda^3 - \frac{1}{2}\left(u - \frac{1}{4}v^2\right)\lambda^2 + \left(\frac{1}{2}uv + \frac{1}{4}v^3 + c_2\right)\lambda - \frac{5}{64}v^4 + \frac{1}{4}u^2 - \frac{1}{16}v_x^2 + c_1 - \frac{1}{2}c_2v.
$$

Foliation for $m = 1$

For the case $m = 1$, putting $h_0 = c_1$ and $h_3 = c_2$ we obtain the foliation of $\mathcal{M}_2$ into leaves $\mathcal{M}_{2,1}^{\mathbf{c}}$. Solving these relations with respect to u_{2x} and v_{2x},

$$
u_{2x} = 6c_2v - 2u^2 + \frac{15}{2}uv^2 + \frac{15}{4}v^4 + \frac{6vu_xv_x - 6uv_x^2 + 2u_x^2 - 32c_1}{4u + 3v^2}, \qquad v_{2x} = -4c_2 - 6uv - \frac{5}{2}v^3,
$$

we arrive at the curve (3.31) for the leaves $\mathcal{M}_{2,1}^{\mathbf{c}}$,

$$
\lambda^5 + c_2\lambda^2 + c_1\lambda^{-1} + H_1\lambda + H_2 = \lambda\mu^2, \tag{7.5}
$$

where

$$H_1 = \frac{5}{64}v^4 - \frac{1}{4}u^2 + \frac{1}{16}v_x^2 - \frac{(2u_x + 3vv_x)^2}{8\,(4u + 3v^2)} + \frac{8c_1}{4u + 3v^2} + \frac{1}{2}c_2 v,$$
$$H_2 = \frac{1}{32}v(v^2 + 2u)\left(4u + 3v^2\right) - \frac{(2u_x + 3vv_x)(vu_x - 2uv_x)}{8\,(4u + 3v^2)} + \frac{4c_1 v}{4u + 3v^2} + \frac{1}{2}c_2 u + \frac{3}{8}c_2 v^2.$$

The Lax matrices $\mathbb{V}_k^{(1)}$ in (3.25) are as follows, $\mathbb{V}_1^{(1)} = \mathbb{V}_1$,

$$\mathbb{V}_2^{(1)} = \begin{pmatrix} -\frac{1}{4}v_x & \frac{1}{2}v + \lambda \\ \lambda^3 - \frac{1}{2}v\lambda^2 - \left(u + \frac{1}{2}v^2\right)\lambda + uv + \frac{5}{8}v^3 + c_2 & \frac{1}{4}v_x \end{pmatrix}$$

$$\mathbb{V}_3^{(1)} = \begin{pmatrix} -\frac{1}{4}v_x\lambda - \frac{1}{4}u_x - \frac{3}{8}vv_x & \lambda^2 + \frac{1}{2}v\lambda + \frac{3}{8}v^2 + \frac{1}{2}u \\ \left(\mathbb{V}_3^{(1)}\right)_{21} & \frac{1}{4}v_x\lambda + \frac{1}{4}u_x + \frac{3}{8}vv_x \end{pmatrix},$$

where

$$\left(\mathbb{V}_3^{(1)}\right)_{21} = \lambda^4 - \frac{1}{2}v\lambda^3 - \frac{1}{2}\left(u - \frac{1}{4}v^2\right)\lambda^2 + \left(\frac{1}{2}uv + \frac{1}{4}v^3 + c_2\right)\lambda - \frac{9v^2v_x^2 + 12vu_xv_x + 4u_x^2 - 64c_1}{8(3v^2 + 4u)}.$$

Foliation for $m = 2$

Finally, for $m = 2$ we put $h_0 = c_1$ and $h_1 = c_2$ which leads to the foliation $\mathcal{M}_{2,2}^{\boldsymbol{c}}$ of $\mathcal{M}_2$. Solving these relations again with respect to u_{2x} and v_{2x},

$$u_{2x} = \frac{48c_2 v}{4u + 3v^2} - \frac{2\left(4u + 9v^2\right)\left(16c_1 + 3uv_x^2 - u_x^2\right) - 36v^3u_xv_x}{\left(4u + 3v^2\right)^2} - 2u^2 + \frac{9}{2}uv^2 + \frac{9}{4}v^4,$$
$$v_{2x} = \frac{256c_1 v - 12v^2u_xv_x - 16vu_x^2 + 48uvv_x^2 + 16uu_xv_x}{\left(4u + 3v^2\right)^2} - \frac{32c_2}{4u + 3v^2} - 4uv - \frac{3}{2}v^3,$$

we arrive at the curve (3.31) for the leaves $\mathcal{M}_{2,2}^{\boldsymbol{c}}$. It has the form

$$\lambda^4 + c_2\lambda^{-1} + c_1\lambda^{-2} + H_1\lambda + H_2 = \lambda^2\mu^2$$

while

$$H_1 = -\frac{(2uv_x - vu_x)\,(2u_x + 3vv_x)}{\left(4u + 3v^2\right)^2} - \frac{1}{4}v\left(2u + v^2\right) + \frac{8c_2}{4u + 3v^2} - \frac{32c_1 v}{\left(4u + 3v^2\right)^2},$$
$$H_2 = \frac{\left(\left(3v^2 - 4u\right)v_x + 4vu_x\right)^2}{16\left(4u + 3v^2\right)^2} - \frac{\left(2u_x + 3vv_x\right)^2}{8\,(4u + 3v^2)} - \frac{1}{64}\left(4u + v^2\right)\left(4u + 3v^2\right) + \frac{4c_2 v}{4u + 3v^2} + \frac{8c_1\left(4u + v^2\right)}{\left(4u + 3v^2\right)^2}.$$

The Lax matrices $\mathbb{V}_k^{(2)}$ in (3.25) are as follows, $\mathbb{V}_1^{(2)} = \mathbb{V}_1$,

$$\mathbb{V}_2^{(2)} = \begin{pmatrix} -\frac{1}{4}v_x & \lambda + \frac{1}{2}v \\ \lambda^3 - \frac{1}{2}v\lambda^2 - \left(u + \frac{1}{2}v^2\right)\lambda - \boldsymbol{\kappa} + \frac{1}{8}v(3v^2 + 4u) & \frac{1}{4}v_x \end{pmatrix},$$

$$\mathbb{V}_3^{(2)} = \begin{pmatrix} -\frac{1}{4}v_x\lambda - \frac{1}{4}u_x - \frac{3}{8}vv_x & \lambda^2 + \frac{1}{2}v\lambda + \frac{3}{8}v^2 + \frac{1}{2}u \\ \lambda^4 - \frac{1}{2}v\lambda^3 - \frac{1}{2}\left(u + \frac{1}{4}v^2\right)\lambda^2 - \boldsymbol{\kappa}\,\lambda - \frac{1}{8}\frac{(3v\,v_x + 2u_x)^2}{3v^2 + 4u} + c_1\frac{8}{3v^2 + 4u} & \frac{1}{4}v_x\lambda + \frac{1}{4}u_x + \frac{3}{8}vv_x \end{pmatrix},$$

where

$$\boldsymbol{\kappa} = \frac{(3v\,v_x + 2u_x)(2uv_x - vu_x)}{(3v^2 + 4u)^2} + c_1\frac{32v}{(3v^2 + 4u)^2} - c_2\frac{8}{(3v^2 + 4u)}.$$

Stäckel system for $m = 0$

Take first $m = 0$. The separation curve (7.4)

$$\lambda^6 + c_2\lambda^3 + c_1\lambda^2 + H_1\lambda + H_2 = \mu^2$$

yields the following Stäckel Hamiltonians in Viéte coordinates

$$H_1 = p_2^2 q_1 + 2p_1 p_2 + q_1^5 - 4q_2 q_1^3 + 3q_2^2 q_1 - c_2 q_1^2 + c_1 q_1 + c_2 q_2, \tag{7.6a}$$
$$H_2 = p_2^2 q_1^2 + 2p_1 p_2 q_1 - p_2^2 q_2 + p_1^2 + q_2 q_1^4 - 3q_2^2 q_1^2 + q_2^3 - c_2 q_2 q_1 + c_1 q_2, \tag{7.6b}$$

that in turn lead to the following Stäckel system

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} 2p_2 \\ 2p_2 q_1 + 2p_1 \\ 2c_1 q_1 - c_1 - p_2^2 - 5q_1^4 + 12q_2 q_1^2 - 3q_2^2 \\ -c_2 + 4q_1^3 - 6q_2 q_1 \end{pmatrix}, \tag{7.7}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_2} = \begin{pmatrix} 2p_2 q_1 + 2p_1 \\ 2p_2 q_1^2 + 2p_1 q_1 - 2p_2 q_2 \\ c_2 q_2 - 2p_2^2 q_1 - 2p_1 p_2 - 4q_2 q_1^3 + 6q_2^2 q_1 \\ c_2 q_1 - c_1 + p_2^2 - q_1^4 + 6q_2 q_1^2 - 3q_2^2 \end{pmatrix}. \tag{7.8}$$

The map (5.4) attains the form given by

$$q_1 = \frac{1}{2}v, \qquad q_2 = \frac{1}{2}u + \frac{3}{8}v^2, \qquad p_1 = \frac{1}{4}u_x + \frac{1}{4}vv_x, \qquad p_2 = \frac{1}{4}v_x, \tag{7.9a}$$

and by

$$c_1 = h_2 \equiv -\frac{9}{8}uv^2 - \frac{1}{4}u_{2x} - \frac{3}{4}u^2 - \frac{1}{2}vv_{2x} - \frac{5}{16}v_x^2 - \frac{15}{64}v^4, \qquad c_2 = h_3 \equiv -\frac{3}{2}uv - \frac{1}{4}v_{2x} - \frac{5}{8}v^3, \tag{7.9b}$$

It maps the first two flows in (7.3) onto the first two components in (7.7) and (7.8), respectively. The remaining two components become identities on $\mathcal{M}_2$ due to the second part of the map (7.9). The Lax matrices $\mathbb{V}_3^{(0)}$, $\mathbb{V}_1^{(0)}$ and $\mathbb{V}_2^{(0)}$ transforms by (7.9) respectively onto $\mathbb{L}$, $\mathbb{U}_1$ and $\mathbb{U}_2$ given by

$$\mathbb{L} = \begin{pmatrix} -p_2\lambda - p_1 - p_2 q_1 & \lambda^2 + q_1\lambda + q_2 \\ \lambda^4 - q_1\lambda^3 + \left(q_1^2 - q_2\right)\lambda^2 + \left(-q_1^3 + c_2 + 2q_1 q_2\right)\lambda + \boldsymbol{\kappa} & p_2\lambda + p_1 + p_2 q_1 \end{pmatrix},$$

$$\mathbb{U}_1 = \begin{pmatrix} 0 & 1 \\ \lambda^2 - 2q_1\lambda + 3q_1^2 - 2q_2 & 0 \end{pmatrix}, \qquad \mathbb{U}_2 = \begin{pmatrix} -p_2 & \lambda + q_1 \\ \lambda^3 - q_1\lambda^2 + \left(q_1^2 - 2q_2\right)\lambda - q_1^3 + 4q_1 q_2 + c_2 & p_2 \end{pmatrix},$$

where

$$\boldsymbol{\kappa} = q_1^4 - p_2^2 + q_2^2 - 3q_1^2 q_2 + c_1 - c_2 q_1.$$

The inverse of the map (7.9) is given by

$$u = 2q_2 - 3q_1^2, \qquad u_x = 4p_1 - 8p_2 q_1, \qquad v = 2q_1, \qquad v_x = 4p_2$$

and

$$u_{2x} = 16c_2 q_1 - 4c_1 - 20p_2^2 - 52q_1^4 + 96q_2 q_1^2 - 12q_2^2, \qquad v_{2x} = -4c_2 + 16q_1^3 - 24q_2 q_1.$$

Stäckel system for $m = 1$

For $m = 1$ the curve (7.5) is

$$\lambda^5 + c_2\lambda^2 + c_1\lambda^{-1} + H_1\lambda + H_2 = \lambda\mu^2$$

yielding the following Stäckel Hamiltonians in Viéte coordinates

$$H_1 = -p_2^2 q_2 + p_1^2 - q_1^4 + 3q_2 q_1^2 - q_2^2 + c_1 \frac{1}{q_2} + c_2 q_1, \tag{7.10a}$$

$$H_2 = -p_2^2 q_2 q_1 - 2p_1 p_2 q_2 - q_2 q_1^3 + 2q_2^2 q_1 + c_1 \frac{q_1}{q_2} + c_2 q_2, \tag{7.10b}$$

giving rise to the following Stäckel system

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} 2p_1 \\ -2p_2 q_2 \\ 4q_1^3 - 6q_2 q_1 - c_2 \\ p_2^2 - 3q_1^2 + 2q_2 + c_1 \frac{1}{q_2^2} \end{pmatrix}, \tag{7.11}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_2} = \begin{pmatrix} -2p_2 q_2 \\ -2p_1 q_2 - 2p_2 q_1 q_2 \\ p_2^2 q_2 - 2q_2^2 + 3q_1^2 q_2 - c_1 \frac{1}{q_2} \\ p_2^2 q_1 + 2p_1 p_2 + q_1^3 - 4q_2 q_1 + c_1 \frac{q_1}{q_2^2} - c_2 \end{pmatrix}. \tag{7.12}$$

The map (5.4) is given by

$$q_1 = \frac{1}{2} v, \qquad q_2 = \frac{1}{2} u + \frac{3}{8} v^2, \qquad p_1 = \frac{1}{4} v_x, \qquad p_2 = -\frac{2u_x + 3vv_x}{4u + 3v^2}, \tag{7.13a}$$

together with

$$\begin{aligned} c_1 = h_0 \equiv & -\frac{3}{32} v^2 u_{2x} + \frac{3}{16} v u_x v_x - \frac{3}{16} u v v_{2x} - \frac{3}{16} u v_x^2 - \frac{9}{64} u v^4 - \frac{3}{8} u^2 v^2 + \frac{1}{16} u_x^2 \\ & - \frac{1}{8} u u_{2x} - \frac{1}{4} u^3 - \frac{9}{64} v^3 v_{2x}, \\ c_2 = h_3 \equiv & -\frac{3}{2} u v - \frac{1}{4} v_{2x} - \frac{5}{8} v^3, \end{aligned} \tag{7.13b}$$

and it maps the first two flows in (7.3) onto the first two components of (7.11) and (7.12), respectively. The remaining two components become identities on $\mathcal{M}_2$ due to (7.13). The Lax matrices $\mathbb{V}_3^{(1)}$, $\mathbb{V}_1^{(1)}$ and $\mathbb{V}_2^{(1)}$ transforms by (7.13) respectively onto $\mathbb{L}$, $\mathbb{U}_1$ and $\mathbb{U}_2$ given by

$$\mathbb{L} = \begin{pmatrix} p_2 q_2 - p_1 \lambda & \lambda^2 + q_1 \lambda + q_2 \\ \lambda^4 - q_1 \lambda^3 + \left(q_1^2 - q_2\right) \lambda^2 - \left(q_1^3 - c_2 - 2q_1 q_2\right) \lambda - p_2^2 q_2 + \frac{c_1}{q_2} & \lambda p_1 - p_2 q_2 \end{pmatrix},$$

$$\mathbb{U}_1 = \begin{pmatrix} 0 & 1 \\ \lambda^2 - 2q_1 \lambda + 3q_1^2 - 2q_2 & 0 \end{pmatrix}, \qquad \mathbb{U}_2 = \begin{pmatrix} -p_1 & \lambda + q_1 \\ \lambda^3 - q_1 \lambda^2 + \left(q_1^2 - 2q_2\right) \lambda - q_1^3 + c_2 + 4q_1 q_2 & p_1 \end{pmatrix}.$$

The inverse of the map (7.13) is given by

$$u = 2q_2 - 3q_1^2, \qquad u_x = -12p_1 q_1 - 4p_2 q_2, \qquad v = 2q_1, \qquad v_x = 4p_1$$

together with

$$u_{2x} = 12c_2 q_1 - \frac{4c_1}{q_2} + 4p_2^2 q_2 - 24p_1^2 - 48q_1^4 + 84q_2 q_1^2 - 8q_2^2, \qquad v_{2x} = -4c_2 + 16q_1^3 - 24q_2 q_1.$$

Stäckel system for $m = 2$

Finally, for $m = 2$ the curve (7.5) is

$$\lambda^4 + c_2 \lambda^{-1} + c_1 \lambda^{-2} + H_1 \lambda + H_2 = \lambda^2 \mu^2,$$

yielding the Stäckel Hamiltonians

$$H_1 = -p_1^2 q_1 - 2p_1p_2q_2 + q_1^3 - 2q_2q_1 - c_1\frac{q_1}{q_2^2} + c_2\frac{1}{q_2}, \tag{7.14a}$$

$$H_2 = -q_2p_1^2 + p_2^2q_2^2 - q_2^2 + q_1^2q_2 + c_1\left(\frac{1}{q_2} - \frac{q_1^2}{q_2^2}\right) + c_2\frac{q_1}{q_2}, \tag{7.14b}$$

that generate the following Stäckel system:

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} -2p_1q_1 - 2p_2q_2 \\ -2p_1q_2 \\ p_1^2 - 3q_1^2 + 2q_2 + c_1\frac{1}{q_2^2} \\ 2p_1p_2 + 2q_1 + c_2\frac{1}{q_2^2} - 2c_1\frac{q_1}{q_2^3} \end{pmatrix}, \tag{7.15}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_2} = \begin{pmatrix} -2p_1q_2 \\ 2p_2q_2^2 \\ -2q_1q_2 + 2c_1\frac{q_1}{q_2^2} - c_2\frac{1}{q_2} \\ -2p_2^2q_2 + p_1^2 - q_1^2 + 2q_2 + c_1\left(\frac{1}{q_2^2} - 2\frac{q_1^2}{q_2^3}\right) + c_2\frac{q_1}{q_2^2} \end{pmatrix}. \tag{7.16}$$

The map (5.4) is given by

$$q_1 = \frac{1}{2}v, \qquad q_2 = \frac{1}{2}u + \frac{3}{8}v^2, \qquad p_1 = -\frac{2u_x + 3vv_x}{4u + 3v^2}, \qquad p_2 = -\frac{2\left(-4vu_x + 4uv_x - 3v^2v_x\right)}{\left(4u+3v^2\right)^2}, \tag{7.17a}$$

and by

$$\begin{aligned} c_1 = h_0 \equiv& -\frac{3}{32}v^2u_{2x} + \frac{3}{16}vu_xv_x - \frac{3}{16}uvv_{2x} - \frac{3}{16}uv_x^2 - \frac{9}{64}uv^4 - \frac{3}{8}u^2v^2 + \frac{1}{16}u_x^2 \\ & - \frac{1}{8}uu_{2x} - \frac{1}{4}u^3 - \frac{9}{64}v^3v_{2x}, \\ c_2 = h_1 \equiv& -\frac{1}{8}vu_{2x} + \frac{1}{8}u_xv_x - \frac{1}{8}uv_{2x} - \frac{3}{4}uv^3 - \frac{3}{4}u^2v - \frac{9}{32}v^2v_{2x} - \frac{9}{64}v^5, \end{aligned} \tag{7.17b}$$

and it maps the first two flows in (7.3) onto the first two components of (7.15) and (7.16), respectively. The remaining two components become identities on $\mathcal{M}_2$ due to (7.17). The Lax matrices $\mathbb{V}_3^{(2)}$, $\mathbb{V}_1^{(2)}$ and $\mathbb{V}_2^{(2)}$ transforms by (7.13) respectively onto $\mathbb{L}$, $\mathbb{U}_1$ and $\mathbb{U}_2$ given by

$$\mathbb{L} = \begin{pmatrix} \left(q_1p_1 + q_2p_2\right)\lambda + q_2p_1 & \lambda^2 + q_1\lambda + q_2 \\ \lambda^4 - q_1\lambda^3 + \left(q_1^2 - q_2\right)\lambda^2 - \left(q_1p_1^2 + 2q_2p_1p_2 - c_2\frac{1}{q_2} + c_1\frac{q_1}{q_2^2}\right)\lambda + \boldsymbol{\kappa} & -\left(q_1p_1 + q_2p_2\right)\lambda - q_2p_1 \end{pmatrix},$$

$$\mathbb{U}_1 = \begin{pmatrix} 0 & 1 \\ \lambda^2 - 2q_1\lambda + 3q_1^2 - 2q_2 & 0 \end{pmatrix},$$

$$\mathbb{U}_2 = \begin{pmatrix} q_1p_1 + q_2p_2 & \lambda + q_1 \\ \lambda^3 - q_1\lambda^2 + \left(q_1^2 - 2q_2\right)\lambda - \left(q_1p_1^2 + 2q_2p_1p_2 - 2q_1q_2 - c_2\frac{1}{q_2} + c_1\frac{q_1}{q_2^2}\right) & -q_1p_1 - q_2p_2 \end{pmatrix},$$

where

$$\boldsymbol{\kappa} = -q_2p_1^2 + c_1\frac{1}{q_2}.$$

The inverse of the map (7.17) is given by

$$u = 2q_2 - 3q_1^2, \qquad u_x = 12p_1q_1^2 + 12p_2q_2q_1 - 4p_1q_2, \qquad v = 2q_1, \qquad v_x = -4p_1q_1 - 4p_2q_2,$$

together with

$$
\begin{aligned}
u_{2x} &= 12c_2q_1 - \frac{4c_1}{q_2} - 24p_1^2q_1^2 - 48p_1p_2q_2q_1 - 24p_2^2q_2^2 + 4p_1^2q_2 - 48q_1^4 + 84q_2q_1^2 - 8q_2^2,\\
v_{2x} &= -4c_2 + 16q_1^3 - 24q_2q_1.
\end{aligned}
$$

Miura maps

Consider again the case $N = n = 2$. For $m = 1$ the Miura map (6.3) between parametrizations $(q_1, q_2, p_1, p_2, c_1, c_2)$ and $(\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2, \bar{c}_1, \bar{c}_2)$ of the stationary manifold $\mathcal{M}_6$ attains the form

$$
\begin{aligned}
&q_1 = \bar{q}_1, \qquad q_2 = \bar{q}_2, \qquad p_1 = -\bar{q}_1\bar{p}_1 - \bar{q}_2\bar{p}_2, \qquad p_2 = \bar{p}_1,\\
&c_1 = \bar{H}_1(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{c}}) = -\bar{p}_2^2\bar{q}_2 + \bar{p}_1^2 - \bar{q}_1^4 + 3\bar{q}_2\bar{q}_1^2 - \bar{q}_2^2 + \bar{c}_1\frac{1}{\bar{q}_2} + \bar{c}_2\bar{q}_1,\\
&c_2 = \bar{c}_2,
\end{aligned}
\tag{7.18}
$$

where $\bar{H}_1$ is given by (7.10) and for $m = 2$ the respective form

$$
\begin{aligned}
&q_1 = \bar{q}_1, \qquad q_2 = \bar{q}_2, \qquad p_1 = \left(\bar{q}_1^2 - \bar{q}_2\right)\bar{p}_1 + \bar{q}_1\bar{q}_2\bar{p}_2, \qquad p_2 = -\bar{q}_1\bar{p}_1 - \bar{q}_2\bar{p}_2,\\
&c_1 = \bar{H}_2(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{c}}) = -\bar{q}_2\bar{p}_1^2 + \bar{p}_2^2\bar{q}_2^2 - \bar{q}_2^2 + \bar{q}_1^2\bar{q}_2 + \bar{c}_1\left(\frac{1}{\bar{q}_2} - \frac{\bar{q}_1^2}{\bar{q}_2^2}\right) + \bar{c}_2\frac{\bar{q}_1}{\bar{q}_2},\\
&c_2 = \bar{H}_1(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{c}}) = -\bar{p}_1^2\bar{q}_1 - 2\bar{p}_1\bar{p}_2\bar{q}_2 + \bar{q}_1^3 - 2\bar{q}_2\bar{q}_1 - \bar{c}_1\frac{\bar{q}_1}{\bar{q}_2^2} + \bar{c}_2\frac{1}{\bar{q}_2}.
\end{aligned}
\tag{7.19}
$$

where $\bar{H}_1$ and $\bar{H}_2$ are given by (7.14). This yields the three-Hamiltonian representation of all the vector fields of the Stäckel system (6.1) (and thus the three-Hamiltonian representation of the stationary cKdV system (3.3)) in the $(q_1, q_2, p_1, p_2, c_1, c_2)$ parametrization:

$$
\begin{aligned}
\pi_0 dH_1 &= \pi_1 dc_1 = \pi_2 dc_2,\\
\pi_0 dH_2 &= \pi_1 dH_1 = \pi_2 dc_1,
\end{aligned}
$$

where $\pi_0 = \pi$, while π_1 is generated by the Miura map (7.18) and π_2 is generated by the Miura map (7.19). Explicitly, the matrix representations of Poisson bi-vectors π_i are as follows

$$
\pi_0 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \qquad
\pi_1 = \begin{pmatrix}
0 & 0 & -q_1 & 1 & \frac{\partial H_1}{\partial p_1} & 0\\
0 & 0 & -q_2 & 0 & \frac{\partial H_1}{\partial p_2} & 0\\
q_1 & q_2 & 0 & -p_2 & -\frac{\partial H_1}{\partial q_1} & 0\\
-1 & 0 & p_2 & 0 & -\frac{\partial H_1}{\partial q_2} & 0\\
-\frac{\partial H_1}{\partial p_1} & -\frac{\partial H_1}{\partial p_2} & \frac{\partial H_1}{\partial q_1} & \frac{\partial H_1}{\partial q_2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\pi_2 = \begin{pmatrix}
0 & 0 & q_1^2 - q_2 & -q_1 & \frac{\partial H_2}{\partial p_1} & \frac{\partial H_1}{\partial p_1}\\
0 & 0 & q_1q_2 & -q_2 & \frac{\partial H_2}{\partial p_2} & \frac{\partial H_1}{\partial p_2}\\
-q_1^2 + q_2 & -q_1q_2 & 0 & q_1p_2 & -\frac{\partial H_2}{\partial q_1} & -\frac{\partial H_1}{\partial q_1}\\
q_1 & q_2 & -q_1p_2 & 0 & -\frac{\partial H_2}{\partial q_2} & -\frac{\partial H_1}{\partial q_2}\\
-\frac{\partial H_2}{\partial p_1} & -\frac{\partial H_2}{\partial p_2} & \frac{\partial H_2}{\partial q_1} & \frac{\partial H_2}{\partial q_2} & 0 & 0\\
-\frac{\partial H_1}{\partial p_1} & -\frac{\partial H_1}{\partial p_2} & \frac{\partial H_1}{\partial q_1} & \frac{\partial H_1}{\partial q_2} & 0 & 0
\end{pmatrix},
$$

where the Hamiltonians H_1, H_2 are given by (7.6). Now, the three Hamiltonian representations for the Stäckel systems associated with $m = 1$ and $m = 2$ can be obtained by means of the Miura maps (7.18) and (7.19).

Let us finally observe that this system has also been discussed in [23], where the authors considered its three equivalent Ostrogradsky representations related with three different albeit equivalent Lagrangian formulations and where the above Poisson operators have also been presented.

7.1.2 The stationary reduction with $n = 3$ and $m = 0$

Here we consider the $n = 3$ stationary DWW system (see Section 7.1), i.e. the case $N = 2$ and $n = 3$.

Stationary system

The third ($n = 3$) DWW stationary system consists of first three evolution equations from the hierarchy (given already in (7.1)):

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \boldsymbol{K}_1 \equiv \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \boldsymbol{K}_2 \equiv \begin{pmatrix} \frac{1}{2}vu_x + uv_x + \frac{1}{4}v_{3x} \\ u_x + \frac{3}{2}vv_x \end{pmatrix}, \tag{7.20}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_3} = \boldsymbol{K}_3 \equiv \begin{pmatrix} \frac{3}{8}v^2u_x + \frac{3}{2}uvv_x + \frac{3}{2}uu_x + \frac{1}{4}u_{3x} + \frac{3}{8}vv_{3x} + \frac{9}{8}v_xv_{2x} \\ \frac{3}{2}vu_x + \frac{3}{2}uv_x + \frac{15}{8}v^2v_x + \frac{1}{4}v_{3x} \end{pmatrix}, \tag{7.21}$$

and the fourth stationary flow, $\boldsymbol{K}_4 = 0$, (cf. and (7.1c)) which in the normal form is given by

$$\begin{aligned} u_{3x} &= -\frac{15}{2}v^2u_x - 15uvv_x - 6uu_x - \frac{1}{4}35v^3v_x - \frac{5}{2}vv_{3x} - 5v_xv_{2x}, \\ v_{5x} &= -24u^2v_x + 40v^3u_x + 60uv^2v_x - 18u_{2x}v_x - 20u_xv_{2x} - 10uv_{3x} + \frac{105}{2}v^4v_x + \frac{15}{2}v^2v_{3x} \\ &\quad - 15vv_xv_{2x} - 15v_x^3. \end{aligned} \tag{7.22}$$

The constraints (7.22) define the $2n + N = 8$-th dimensional stationary manifold $\mathcal{M}_3$ parametrized by the jet coordinates $[\boldsymbol{u}] = (u, u_x, u_{2x}, v, v_x \ldots, v_{4x})$. The two vector fields (7.20) preserve their form in the above parametrization of $\mathcal{M}_3$, while the vector field (7.21) attains on $\mathcal{M}_3$, parametrized as above, the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_3} = \begin{pmatrix} -\frac{3}{2}v^2u_x - \frac{9}{4}uvv_x - \frac{35}{16}v^3v_x - \frac{1}{4}vv_{3x} - \frac{1}{8}v_xv_{2x} \\ \frac{3vu_x}{2}vu_x + \frac{3}{2}uv_x + \frac{15}{8}v^2v_x + \frac{1}{4}v_{3x} \end{pmatrix}. \tag{7.23}$$

The corresponding spectral curve (3.29) is

$$\lambda^8 + h_4\lambda^4 + h_3\lambda^3 + h_2\lambda^2 + h_1\lambda + h_0 = \mu^2. \tag{7.24}$$

The corresponding functions h_r in (3.13) attain the explicit form

$$\begin{aligned} h_4 &= -\frac{35}{64}v^4 - \frac{15}{8}uv^2 - \frac{5}{8}v_{2x}v - \frac{3}{4}u^2 - \frac{5}{16}v_x^2 - \frac{1}{4}u_{2x}, \\ h_3 &= -\frac{7}{32}v^5 - \frac{5}{4}uv^3 - \frac{25}{32}v_{2x}v^2 - \frac{3}{2}u^2v - \frac{15}{16}v_x^2v - \frac{1}{2}u_{2x}v - \frac{5}{8}u_xv_x - \frac{5}{8}uv_{2x} - \frac{1}{16}v_{4x}, \\ h_2 &= -\frac{35}{256}v^6 - \frac{55}{64}uv^4 - \frac{15}{32}v_{2x}v^3 - \frac{21}{16}u^2v^2 - \frac{15}{64}v_x^2v^2 - \frac{9}{32}u_{2x}v^2 - \frac{5}{8}uv_{2x}v - \frac{1}{32}v_{4x}v - \frac{1}{4}u^3 + \frac{1}{16}u_x^2 \\ &\quad - \frac{1}{64}v_{2x}^2 - \frac{1}{8}uu_{2x} + \frac{1}{32}v_xv_{3x}, \\ h_1 &= -\frac{25}{256}v^7 - \frac{45}{64}uv^5 - \frac{95}{256}v_{2x}v^4 - \frac{23}{16}u^2v^3 - \frac{15}{128}v_x^2v^3 - \frac{7}{32}u_{2x}v^3 + \frac{15}{64}u_xv_xv^2 - \frac{15}{16}uv_{2x}v^2 \\ &\quad - \frac{3}{128}v_{4x}v^2 - \frac{3}{4}u^3v + \frac{3}{16}u_x^2v - \frac{15}{32}uv_x^2v - \frac{1}{16}v_{2x}^2v - \frac{3}{8}uu_{2x}v + \frac{3}{64}v_xv_{3x}v - \frac{3}{16}uu_xv_x - \frac{5}{16}u^2v_{2x} \\ &\quad - \frac{3}{64}v_x^2v_{2x} - \frac{1}{32}u_{2x}v_{2x} + \frac{1}{32}u_xv_{3x} - \frac{1}{32}uv_{4x}, \\ h_0 &= -\frac{25}{256}uv^6 - \frac{75}{512}v_{2x}v^5 - \frac{15}{32}u^2v^4 - \frac{75}{1024}v_x^2v^4 - \frac{15}{128}u_{2x}v^4 + \frac{15}{128}u_xv_xv^3 - \frac{35}{64}uv_{2x}v^3 - \frac{5}{256}v_{4x}v^3 \\ &\quad - \frac{9}{16}u^3v^2 + \frac{9}{64}u_x^2v^2 - \frac{45}{128}uv_x^2v^2 - \frac{15}{256}v_{2x}^2v^2 - \frac{9}{32}uu_{2x}v^2 + \frac{15}{256}v_xv_{3x}v^2 - \frac{9}{32}uu_xv_xv \\ &\quad - \frac{15}{32}u^2v_{2x}v - \frac{15}{128}v_x^2v_{2x}v - \frac{3}{64}u_{2x}v_{2x}v + \frac{3}{64}u_xv_{3x}v - \frac{3}{64}uv_{4x}v + \frac{9}{64}u^2v_x^2 - \frac{1}{16}uv_{2x}^2 + \frac{1}{256}v_{3x}^2 \\ &\quad - \frac{3}{32}u_xv_xv_{2x} + \frac{3}{64}uv_xv_{3x} - \frac{1}{128}v_{2x}v_{4x}. \end{aligned}$$

Foliation for $m = 0$

We will now consider the Hamiltonian foliation of $\mathcal{M}_3$ defined by (3.16) for $m = 0$, that is $h_4 = c_2$ and $h_3 = c_1$, which yields the $2n = 6$-dimensional leaves $\mathcal{M}^{\mathfrak{c}}_{3,0}$ defined by the conditions

$$\begin{aligned} u_{2x} &= -4c_2 - 3u^2 - \frac{15}{2}uv^2 - \frac{35}{16}v^4 - \frac{5}{2}vv_{2x} - \frac{5}{4}v_x^2, \\ v_{4x} &= 32c_2 v - 16c_1 + 40uv^3 - 10u_x v_x - 10uv_{2x} + 14v^5 + \frac{15}{2}v^2 v_{2x} - 5vv_x^2. \end{aligned} \tag{7.25}$$

Each leaf $\mathcal{M}^{\mathfrak{c}}_{3,0}$ is now parametrized by the jet coordinates $[\boldsymbol{u}] = (u, u_x, v, v_x, v_{2x}, v_{3x})$. All three vector fields $\boldsymbol{K}_i$ preserve their explicit form when reducing from $\mathcal{M}_3$ to the leaf $\mathcal{M}^{\mathfrak{c}}_{3,0}$, due to the fact that they do not contain neither u_{2x} nor v_{4x} that define this leaf within the manifold $\mathcal{M}_3$. Taking into account conditions (7.25) the spectral curve (7.24) attains the form (cf. (3.31))

$$\lambda^8 + c_2\lambda^4 + c_1\lambda^3 + H_1\lambda^2 + H_2\lambda + H_3 = \mu^2, \tag{7.26}$$

while H_i are explicitly given by

$$\begin{aligned} H_1 &= \frac{1}{16}u_x^2 + \frac{5}{16}vv_x u_x + \frac{35}{128}v^2v_x^2 + \frac{5}{32}uv_x^2 + \frac{1}{32}v_x v_{3x} + \frac{1}{8}u^3 + \frac{15}{32}u^2v^2 + \frac{35}{128}uv^4 + \frac{21}{512}v^6 - \frac{1}{64}v_{2x}^2 \\ &\quad + \frac{1}{2}c_2 u + \frac{1}{8}c_2 v^2 + \frac{1}{2}c_1 v, \\ H_2 &= \frac{35}{128}v_x^2v^3 + \frac{15}{32}u_x v_x v^2 + \frac{3}{16}u_x^2 v + \frac{5}{32}uv_x^2 v + \frac{3}{64}v_x v_{3x} v + \frac{1}{8}uu_x v_x - \frac{1}{128}v_x^2 v_{2x} + \frac{1}{32}u_x v_{3x} \\ &\quad + \frac{1}{2}c_2 uv + \frac{1}{8}c_2 v^3 + \frac{1}{8}c_2 v_{2x} + \frac{1}{2}c_1 u + \frac{3}{8}c_1 v^2, \end{aligned}$$

and

$$\begin{aligned} H_3 &= \frac{175}{1024}v_x^2v^4 + \frac{5}{16}u_x v_x v^3 + \frac{9}{64}u_x^2 v^2 + \frac{15}{64}uv_x^2 v^2 + \frac{15}{256}v_x v_{3x} v^2 + \frac{3}{16}uu_x v_x v - \frac{5}{256}v_x^2 v_{2x} v \\ &\quad + \frac{3}{64}u_x v_{3x} v + \frac{9}{64}u^2 v_x^2 + \frac{1}{256}v_{3x}^2 - \frac{1}{64}u_x v_x v_{2x} + \frac{3}{64}uv_x v_{3x} + \frac{9}{32}u^3v^2 + \frac{15}{128}u^2v^4 \\ &\quad + \frac{9}{64}u^2 vv_{2x} - \frac{21}{512}uv^6 + \frac{5}{128}uv^3 v_{2x} + \frac{1}{64}uv_{2x}^2 - \frac{35}{2048}v^8 - \frac{7}{1024}v^5 v_{2x} - \frac{3}{8}c_2 uv^2 \\ &\quad - \frac{1}{32}5c_2 v^4 - \frac{1}{16}c_2 vv_{2x} + \frac{3}{4}c_1 uv + \frac{5}{16}c_1 v^3 + \frac{1}{8}c_1 v_{2x}. \end{aligned}$$

Stäckel system

The relation (7.26), if treated as the separation curve (5.1), yields the Stäckel Hamiltonians (given here directly in Viète coordinates (4.7))

$$\begin{aligned} H_1 &= 2p_3p_2q_1 + p_3^2q_2 + p_2^2 + 2p_1p_3 - q_1^6 + 5q_2q_1^4 - 4q_3q_1^3 - 6q_2^2q_1^2 + 6q_2q_3q_1 + q_2^3 - q_3^2 + c_2q_2 - c_2q_1^2 + c_1q_1, \\ H_2 &= 2p_2^2q_1 + 2p_3p_2q_1^2 + 2p_1p_3q_1 + p_3^2q_1q_2 - p_3^2q_3 + 2p_1p_2 + q_3q_1^4 + 4q_2^2q_1^3 - 6q_2q_3q_1^2 - 3q_2^3q_1 + 2q_3^2q_1, \\ &\quad - q_1^5q_2 + 3q_2^2q_3 + c_2q_3 - c_2q_1q_2 + c_1q_2 \\ H_3 &= 2p_2p_1q_1 + 2p_3p_1q_2 + p_2^2q_1^2 + p_3^2q_2^2 + 2p_2p_3q_1q_2 - 2p_2p_3q_3 - p_3^2q_1q_3 + p_1^2 + 4q_2q_3q_1^3 - 3q_3^2q_1^2 - 3q_2^2q_3q_1 \\ &\quad + 2q_2q_3^2 - q_1^5q_3 - c_2q_1q_3 + c_1q_3, \end{aligned}$$

which generate the following Stäckel system (cf. (4.9)) on $\mathcal{M}^{\mathfrak{c}}_{3,0}$:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}_{t_1} = \begin{pmatrix} 2p_3 \\ 2p_2 + 2p_3q_1 \\ 2p_1 + 2p_2q_1 + 2p_3q_2 \\ 6q_1^5 - 20q_2q_1^3 + 12q_3q_1^2 + 12q_2^2q_1 + 2c_2q_1 - c_1 - 2p_2p_3 - 6q_2q_3 \\ -5q_1^4 + 12q_2q_1^2 - 6q_3q_1 - p_3^2 - 3q_2^2 - c_2 \\ 4q_1^3 - 6q_2q_1 + 2q_3 \end{pmatrix},$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}_{t_2} = \begin{pmatrix} 2p_2 + 2p_3q_1 \\ 2p_3q_1^2 + 4p_2q_1 + 2p_1 \\ 2p_2q_1^2 + 2p_1q_1 + 2p_3q_2q_1 - 2p_3q_3 \\ 5q_2q_1^4 - 4q_3q_1^3 - 12q_2^2q_1^2 - 4p_2p_3q_1 + 12q_2q_3q_1 + 3q_2^3 - 2p_2^2 - 2q_3^2 - 2p_1p_3 - p_3^2q_2 + c_2q_2 \\ q_1^5 - 8q_2q_1^3 + 6q_3q_1^2 - p_3^2q_1 + 9q_2^2q_1 + c_2q_1 - c_1 - 6q_2q_3 \\ -q_1^4 + 6q_2q_1^2 - 4q_3q_1 + p_3^2 - 3q_2^2 - c_2 \end{pmatrix},$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}_{t_3} = \begin{pmatrix} 2p_1 + 2p_2q_1 + 2p_3q_2 \\ 2p_2q_1^2 + 2p_1q_1 + 2p_3q_2q_1 - 2p_3q_3 \\ 2p_3q_2^2 + 2p_1q_2 + 2p_2q_1q_2 - 2p_2q_3 - 2p_3q_1q_3 \\ 5q_3q_1^4 - 12q_2q_3q_1^2 - 2p_2^2q_1 + 6q_3^2q_1 - 2p_1p_2 - 2p_2p_3q_2 + p_3^2q_3 + 3q_2^2q_3 + c_2q_3 \\ -4q_3q_1^3 - 2p_2p_3q_1 + 6q_2q_3q_1 - 2q_3^2 - 2p_1p_3 - 2p_3^2q_2 \\ q_1^5 - 4q_2q_1^3 + 6q_3q_1^2 + p_3^2q_1 + 3q_2^2q_1 + c_2q_1 - c_1 + 2p_2p_3 - 4q_2q_3 \end{pmatrix}.$$

Their Lax representation (4.12) is given by the Lax matrix (4.13), given in Viète coordinates by

$$\mathbb{L} = \begin{pmatrix} -p_3\lambda^2 - (p_2 + p_3q_1)\,\lambda - p_1 - p_2q_1 - p_3q_2 & \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 \\ \mathbb{L}_{21} & p_3\lambda^2 + (p_2 + p_3q_1)\,\lambda + p_1 + p_2q_1 + p_3q_2 \end{pmatrix},$$

where

$$\begin{aligned} \mathbb{L}_{21} = {} & \lambda^5 - q_1\lambda^4 + \left(q_1^2 - q_2\right)\lambda^3 - \left(q_1^3 - 2q_2q_1 + q_3\right)\lambda^2 + \left(q_1^4 - 3q_2q_1^2 + 2q_3q_1 - p_3^2 + q_2^2 + c_2\right)\lambda \\ & - q_1^5 - 3q_1q_2^2 - 2p_2p_3 - p_3^2q_1 + 4q_1^3q_2 - 3q_1^2q_3 + 2q_2q_3 + c_1 - c_2q_1, \end{aligned}$$

and by the auxiliary matrices (4.18), given in Viète coordinates by

$$\mathbb{U}_1 = \begin{pmatrix} 0 & 1 \\ \lambda^2 - 2q_1\lambda + 3q_1^2 - 2q_2 & 0 \end{pmatrix}, \qquad \mathbb{U}_2 = \begin{pmatrix} -p_3 & \lambda + q_1 \\ \lambda^3 - q_1\lambda^2 + \left(q_1^2 - 2q_2\right)\lambda - q_1^3 + 4q_1q_2 - 2q_3 & p_3 \end{pmatrix},$$

$$\mathbb{U}_3 = \begin{pmatrix} -\lambda p_3 - p_2 - p_3q_1 & \lambda^2 + q_1\lambda + q_2 \\ \lambda^4 - q_1\lambda^3 + \left(q_1^2 - q_2\right)\lambda^2 - \left(q_1^3 - 2q_1q_2 + 2q_3\right)\lambda + \boldsymbol{\kappa} & \lambda p_3 + p_2 + p_3q_1 \end{pmatrix},$$

where

$$\boldsymbol{\kappa} = +q_1^4 - p_3^2 + q_2^2 + c_2 - 3q_1^2q_2 + 4q_1q_3.$$

Transformation between jet and canonical coordinates

Finally, let us present the map (5.4) between the jet variables $[\boldsymbol{u}] = (u, u_x, u_{2x}, v, v_x \ldots, v_{4x})$ and the (extended by Casimirs c_i) Viète coordinates on the stationary manifold $\mathcal{M}_3$. It is explicitly given by

$$\begin{aligned} & q_1 = \frac{1}{2}v, \qquad q_2 = \frac{1}{2}u + \frac{3}{8}v^2, \qquad q_3 = \frac{3}{4}uv + \frac{5}{16}v^3 + \frac{1}{8}v_{2x}, \\ & p_1 = \frac{1}{4}vu_x + \frac{1}{4}uv_x + \frac{1}{4}v^2v_x + \frac{1}{16}v_{3x}, \qquad p_2 = \frac{1}{4}u_x + \frac{1}{4}vv_x, \qquad p_3 = \frac{1}{4}v_x \end{aligned} \tag{7.27a}$$

and by

$$\begin{aligned} c_1 = h_3 & \equiv -\frac{7}{32}v^5 - \frac{5}{4}uv^3 - \frac{25}{32}v_{2x}v^2 - \frac{3}{2}u^2v - \frac{15}{16}v_x^2v - \frac{1}{2}u_{2x}v - \frac{5}{8}u_xv_x - \frac{5}{8}uv_{2x} - \frac{1}{16}v_{4x}, \\ c_2 = h_4 & \equiv -\frac{35}{64}v^4 - \frac{15}{8}uv^2 - \frac{5}{8}v_{2x}v - \frac{3}{4}u^2 - \frac{5}{16}v_x^2 - \frac{1}{4}u_{2x}. \end{aligned} \tag{7.27b}$$

Applying the map (7.27a) to the above Stäckel system we reconstruct the respective vector fields (7.20) and (7.23) together with the constraints (7.25) or equivalently (7.27b).

The inverse of the map (7.27) is given by

$$\begin{aligned} & u = 2q_2 - 3q_1^2, \qquad u_x = 4p_2 - 8p_3q_1, \\ & v = 2q_1, \qquad v_x = 4p_3, \qquad v_{2x} = 16q_1^3 - 24q_2q_1 + 8q_3, \qquad v_{3x} = 48p_3q_1^2 - 32p_2q_1 - 32p_3q_2 + 16p_1, \end{aligned}$$

together with

$$
\begin{aligned}
u_{2x} &= -4c_2 - 20p_3^2 - 52q_1^4 + 96q_2q_1^2 - 40q_3q_1 - 12q_2^2,\\
v_{4x} &= 160p_3^2q_1 - 160p_2p_3 + 448q_1^5 - 1120q_2q_1^3 + 480q_3q_1^2 + 480q_2^2q_1 - 160q_2q_3 - 16c_1 + 64c_2q_1.
\end{aligned}
$$

7.2 The case of $N = 4$, $n = 2$ and $m = 0$

Let us now take a closer look at four-component ($N = 4$) stationary cKdV system with $n = 2$. We will again only consider the case $m = 0$.

Four component cKdV hierarchy

Assume thus that $N = 4$ and denote $\boldsymbol{u} = (u_0, u_1, u_2, u_3)^T \equiv (u, v, w, r)^T$. Then, the coefficients P_i of the series (2.6) are

$$
\begin{aligned}
&P_0 = 2, \qquad P_1 = r, \qquad P_2 = \frac{3}{4}r^2 + w, \qquad P_3 = \frac{5}{8}r^3 + \frac{3}{2}rw + v,\\
&P_4 = \frac{35}{64}r^4 + \frac{15}{8}r^2w + \frac{3}{2}rv + u + \frac{3}{4}w^2,\\
&P_5 = \frac{63}{128}r^5 + \frac{35}{16}r^3w + \frac{15}{8}r^2v + \frac{3}{2}ru + \frac{15}{8}rw^2 + \frac{1}{4}r_{2x} + \frac{3}{2}vw\\
&P_6 = \frac{231}{512}r^6 + \frac{315}{128}r^4w + \frac{35}{16}r^3v + \frac{15}{8}r^2u + \frac{105}{32}r^2w^2 + \frac{15}{4}rvw + \frac{5}{8}rr_{2x} + \frac{5}{16}r_x^2 + \frac{3}{2}uw\\
&\qquad + \frac{3}{4}v^2 + \frac{5}{8}w^3 + \frac{1}{4}w_{2x},\\
&\quad \vdots
\end{aligned}
$$

The first three members of the four component cKdV hierarchy (2.9) have the form

$$
\begin{pmatrix} u \\ v \\ w \\ r \end{pmatrix}_{t_1} = \boldsymbol{K}_1 \equiv \begin{pmatrix} u_x \\ v_x \\ w_x \\ r_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \\ w \\ r \end{pmatrix}_{t_2} = \boldsymbol{K}_2 \equiv \begin{pmatrix} ur_x + \frac{1}{2}ru_x + \frac{1}{4}r_{3x} \\ vr_x + \frac{1}{2}rv_x + u_x \\ wr_x + \frac{1}{2}rw_x + v_x \\ \frac{3}{2}rr_x + w_x \end{pmatrix}, \tag{7.28}
$$

$$
\begin{pmatrix} u \\ v \\ w \\ r \end{pmatrix}_{t_3} = \boldsymbol{K}_3 \equiv \begin{pmatrix} \frac{3}{8}r^2u_x + \frac{3}{2}rur_x + \frac{3}{8}rr_{3x} + \frac{9}{8}r_xr_{2x} + \frac{1}{2}wu_x + uw_x + \frac{1}{4}w_{3x} \\ \frac{3}{8}r^2v_x + \frac{1}{2}ru_x + ur_x + \frac{3}{2}rvr_x + \frac{1}{4}r_{3x} + \frac{1}{2}wv_x + vw_x \\ \frac{3}{8}r^2w_x + \frac{1}{2}rv_x + vr_x + \frac{3}{2}rwr_x + u_x + \frac{3}{2}ww_x \\ \frac{15}{8}r^2r_x + \frac{3}{2}rw_x + \frac{3}{2}wr_x + v_x. \end{pmatrix}
$$

Their zero-curvature curvature representation (2.33) is generated by the Lax matrix

$$
\mathbb{V}_1 = \begin{pmatrix} 0 & 1 \\ \lambda^4 - r\lambda^3 - w\lambda^2 - v\lambda - u & 0 \end{pmatrix}
$$

and the auxiliary matrices

$$
\begin{aligned}
\mathbb{V}_2 &= \begin{pmatrix} -\frac{1}{4}r_x & \lambda + \frac{1}{2}r \\ \lambda^5 - \frac{1}{2}r\lambda^4 - \left(\frac{1}{2}r^2 + w\right)\lambda^3 - \left(v + \frac{1}{2}wr\right)\lambda^2 - \left(u + \frac{1}{2}vr\right)\lambda - \frac{1}{2}ur - \frac{1}{4}r_{2x} & \frac{1}{4}r_x \end{pmatrix},\\
\mathbb{V}_3 &= \begin{pmatrix} -\frac{1}{4}r_x\lambda - \frac{1}{4}w_x - \frac{3}{8}rr_x & \lambda^2 + \frac{1}{2}r\lambda + \frac{3}{8}r^2 + \frac{1}{2}w \\ (\mathbb{V}_3)_{21} & \frac{1}{4}r_x\lambda + \frac{1}{4}w_x + \frac{3}{8}rr_x \end{pmatrix},
\end{aligned}
$$

where

$$(\mathbb{V}_3)_{21} = \lambda^6 - \frac{1}{2}r\lambda^5 - \left(\frac{1}{8}r^2 + \frac{1}{2}w\right)\lambda^4 - \left(\frac{3}{8}r^3 + wr + v\right)\lambda^3 - \left(\frac{1}{2}w^2 + \frac{3}{8}r^2w + u + \frac{1}{2}vr\right)\lambda^2$$
$$- \left(\frac{3}{8}vr^2 + \frac{1}{2}ur + \frac{1}{2}vw + \frac{1}{4}r_{2x}\right)\lambda - \frac{3}{8}ur^2 - \frac{3}{8}r_x^2 - \frac{1}{2}uw - \frac{1}{4}w_{2x} - \frac{3}{8}rr_{2x}.$$

Stationary system

In the case of $N = 4$ the second cKdV stationary system consists of the two evolution equations in (7.28) and the third stationary flow, $\boldsymbol{K}_3 = 0$, which in the normal form is given by

$$\begin{aligned}
u_x &= \frac{15}{16}r^3r_x + \frac{3}{8}r^2w_x - vr_x - \frac{3}{4}rwr_x - \frac{3}{2}ww_x, \qquad v_x = -\frac{15}{8}r^2r_x - \frac{3}{2}rw_x - \frac{3}{2}wr_x,\\
w_{3x} &= -\frac{45}{16}r^5r_x - \frac{45}{16}r^4w_x - 12r^3wr_x + \frac{15}{2}r^2vr_x - \frac{15}{2}r^2ww_x + 6rvw_x + 2vwr_x - 3rw^2r_x - \frac{9}{2}r_xr_{2x}\\
&\quad - 4uw_x + 3w^2w_x,\\
r_{3x} &= \frac{15}{16}r^4r_x + \frac{3}{2}r^3w_x + \frac{15}{2}r^2wr_x - 4ur_x - 4rvr_x + 3w^2r_x + 6rww_x - 4vw_x.
\end{aligned}$$

These constraints define the $2n + N = 8$-th dimensional stationary manifold $\mathcal{M}_2$, parametrized by the jet coordinates $[\boldsymbol{u}] = (u, v, w, w_x, w_{2x}, r, r_x, r_{2x})$. The vector fields (7.28) attain on $\mathcal{M}_2$, parametrized as above, the form:

$$\begin{pmatrix} u\\ v\\ w\\ r \end{pmatrix}_{t_1} = \begin{pmatrix} \frac{15}{16}r^3r_x + \frac{3}{8}r^2w_x - vr_x - \frac{3}{4}rwr_x - \frac{3}{2}ww_x\\ -\frac{15}{8}r^2r_x - \frac{3}{2}rw_x - \frac{3}{2}wr_x\\ w_x\\ r_x \end{pmatrix},$$
$$\begin{pmatrix} u\\ v\\ w\\ r \end{pmatrix}_{t_2} = \begin{pmatrix} \frac{45}{64}r^4r_x + \frac{9}{16}r^3w_x + \frac{3}{2}r^2wr_x - \frac{3}{2}rvr_x + \frac{3}{4}w^2r_x + \frac{3}{4}rww_x - vw_x\\ -\frac{3}{8}r^2w_x - \frac{3}{2}rwr_x - \frac{3}{2}ww_x\\ -\frac{15}{8}r^2r_x - rw_x - \frac{1}{2}wr_x\\ \frac{3}{2}rr_x + w_x \end{pmatrix}.$$

The spectral curve (3.29) corresponding to $\mathcal{M}_3$ is

$$\lambda^8 + h_5\lambda^5 + h_4\lambda^4 + h_3\lambda^3 + h_2\lambda^2 + h_1\lambda + h_0 = \mu^2, \tag{7.29}$$

while the functions h_k in (3.13) are

$$\begin{aligned}
h_5 &= -\frac{5}{8}r^3 - \frac{3}{2}wr - v, \qquad h_4 = -\frac{15}{64}r^4 - \frac{9}{8}wr^2 - vr - \frac{3}{4}w^2 - u,\\
h_3 &= -\frac{9}{64}r^5 - \frac{3}{4}wr^3 - vr^2 - \frac{3}{4}w^2r - ur - vw - \frac{1}{4}r_{2x},\\
h_2 &= -\frac{9}{64}wr^4 - \frac{3}{8}vr^3 - \frac{3}{8}w^2r^2 - ur^2 - \frac{1}{2}vwr - \frac{1}{2}r_{2x}r - \frac{1}{4}w^3 - \frac{5}{16}r_x^2 - uw - \frac{1}{4}w_{2x},\\
h_1 &= -\frac{9}{64}vr^4 - \frac{3}{8}ur^3 - \frac{3}{8}vwr^2 - \frac{9}{32}r_{2x}r^2 - \frac{1}{2}uwr - \frac{1}{8}w_{2x}r - \frac{1}{4}vw^2 + \frac{1}{8}w_xr_x - \frac{1}{8}wr_{2x},\\
h_0 &= -\frac{9}{64}ur^4 - \frac{9}{64}r_{2x}r^3 - \frac{3}{8}uwr^2 - \frac{3}{32}w_{2x}r^2 + \frac{3}{16}w_xr_xr - \frac{3}{16}wr_{2x}r - \frac{1}{4}uw^2 + \frac{1}{16}w_x^2\\
&\quad - \frac{3}{16}wr_x^2 - \frac{1}{8}ww_{2x}.
\end{aligned}$$

Foliation

Let us consider the Hamiltonian foliation of $\mathcal{M}_2$ defined by (3.16) for $m = 0$. That is, we consider the foliation of $\mathcal{M}_2$ given by the conditions

$$h_5 = c_4, \quad h_4 = c_3, \quad h_3 = c_2, \quad h_2 = c_1,$$

which in the normal form are given by

$$
\begin{aligned}
u &= c_4 r - c_3 + \frac{25}{64}r^4 + \frac{3}{8}r^2 w - \frac{3}{4}w^2, \qquad v = -c_4 - \frac{5}{8}r^3 - \frac{3}{2}rw,\\
w_{2x} &= -\frac{5}{2}c_4 r^3 - 4c_3 r^2 - 10c_4 rw + 8c_2 r + 4c_3 w - 4c_1 - \frac{11}{8}r^6 - \frac{65}{8}r^4 w - 9r^2w^2 - \frac{5}{4}r_x^2 + 2w^3, \\
r_{2x} &= 4c_3 r + 4c_4 w - 4c_2 + \frac{3}{8}r^5 + 4r^3 w + 6rw^2.
\end{aligned} \tag{7.30}
$$

The above conditions define leaves $\mathcal{M}^{\mathbf{c}}_{2,0}$ parametrized by the jet coordinates $[\boldsymbol{u}] = (w, w_x, r, r_x)$. Notice that the field variables u and v are entirely eliminated using (7.30). In consequence, the vector fields (7.28) on the leave $\mathcal{M}^{\mathbf{c}}_{2,0}$ attains the *two-component* form

$$
\begin{pmatrix} w \\ r \end{pmatrix}_{t_1} = \begin{pmatrix} w_x \\ r_x \end{pmatrix}, \qquad \begin{pmatrix} w \\ r \end{pmatrix}_{t_2} = \begin{pmatrix} -\frac{15}{8}r^2 r_x - r w_x - \frac{1}{2}w r_x \\ \frac{3}{2}r r_x + w_x. \end{pmatrix} \tag{7.31}
$$

as the remaining components in these vector fields become identities.

Taking into account (7.30) the spectral curve (7.29) takes the form, cf. (3.31),

$$
\lambda^8 + c_4\lambda^5 + c_3\lambda^4 + c_2\lambda^3 + c_1\lambda^2 + H_1\lambda + H_2 = \mu^2, \tag{7.32}
$$

where

$$
\begin{aligned}
H_1 &= \frac{1}{8}r_x w_x + \frac{5}{32}r r_x^2 + \frac{1}{128}r^7 - \frac{3}{64}r^5 w - \frac{1}{4}r^3w^2 - \frac{1}{4}rw^3 + \frac{5}{64}c_4 r^4 - \frac{1}{4}c_4 w^2 - \frac{1}{4}c_3 r^3 - \frac{1}{2}c_3 rw\\
&\quad + \frac{1}{8}c_2 r^2 + \frac{1}{2}c_2 w + \frac{1}{2}c_1 r,\\
H_2 &= \frac{15}{128}r^2 r_x^2 - \frac{1}{32}w r_x^2 + \frac{3}{16}r r_x w_x + \frac{1}{16}w_x^2 + \frac{87}{4096}r^8 + \frac{13}{128}r^6 w + \frac{17}{128}r^4 w^2 - \frac{1}{16}w^4 + \frac{3}{32}c_4 r^5\\
&\quad + \frac{5}{16}c_4 r^3 w + \frac{1}{4}c_4 r w^2 - \frac{1}{64}3c_3 r^4 - \frac{1}{4}c_3 r^2 w - \frac{1}{4}c_3 w^2 - \frac{1}{16}3c_2 r^3 - \frac{1}{4}c_2 rw + \frac{3}{8}c_1 r^2 + \frac{1}{2}c_1 w.
\end{aligned}
$$

Stäckel system

The Stäckel Hamiltonians, defined by the separation curve (7.32) and written in Viète coordinates are

$$
\begin{aligned}
H_1 &= q_1p_2^2 + 2p_1p_2 + q_1^7 - 6q_2q_1^5 + 10q_2^2q_1^3 - 4q_2^3q_1 - c_4q_1^4 + 3c_4q_2q_1^2 - c_4q_2^2 + c_3q_1^3 - 2c_3q_1q_2\\
&\quad + c_2q_2 - c_2q_1^2 + c_1q_1,\\
H_2 &= p_1^2 + 2p_2q_1p_1 + p_2^2q_1^2 - p_2^2q_2 + q_2q_1^6 - 5q_2^2q_1^4 + 6q_2^3q_1^2 - q_2^4 + 2c_4q_1q_2^2 - c_4q_1^3q_2 + c_3q_1^2q_2 - c_3q_2^2\\
&\quad - c_2q_1q_2 + c_1q_2.
\end{aligned}
$$

They generate the following Stäckel system

$$
\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} 2p_2 \\ 2p_1 + 2p_2q_1 \\ -7q_1^6 + 30q_2q_1^4 + 4c_4q_1^3 - 30q_2^2q_1^2 - 3c_3q_1^2 + 2c_2q_1 - 6c_4q_2q_1 + 4q_2^3 - p_2^2 - c_1 + 2c_3q_2 \\ 6q_1^5 - 20q_2q_1^3 - 3c_4q_1^2 + 12q_2^2q_1 + 2c_3q_1 - c_2 + 2c_4q_2 \end{pmatrix},
$$

$$
\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_2} = \begin{pmatrix} 2p_1 + 2p_2q_1 \\ 2p_2q_1^2 + 2p_1q_1 - 2p_2q_2 \\ -6q_2q_1^5 + 20q_2^2q_1^3 + 3c_4q_2q_1^2 - 12q_2^3q_1 - 2p_2^2q_1 - 2c_3q_2q_1 - 2c_4q_2^2 - 2p_1p_2 + c_2q_2 \\ -q_1^6 + 10q_2q_1^4 + c_4q_1^3 - 18q_2^2q_1^2 - c_3q_1^2 + c_2q_1 - 4c_4q_2q_1 + 4q_2^3 + p_2^2 - c_1 + 2c_3q_2 \end{pmatrix}.
$$

Their Lax representation (4.12) contains the Lax matrix (4.13), given in Viète coordinates by

$$
\mathbb{L} = \begin{pmatrix} -p_2\lambda - p_1 - p_2q_1 & \lambda^2 + q_1\lambda + q_2 \\ \mathbb{L}_{21} & p_2\lambda + p_1 + p_2q_1 \end{pmatrix},
$$

where

$$\mathbb{L}_{21} = \lambda^6 - q_1\lambda^5 + \left(q_1^2 - q_2\right)\lambda^4 + \left(-q_1^3 + 2q_2q_1 + c_4\right)\lambda^3 + \left(q_1^4 - 3q_2q_1^2 - c_4q_1 + q_2^2 + c_3\right)\lambda^2$$
$$+ \left(-q_1^5 + 4q_2q_1^3 + c_4q_1^2 - 3q_2^2q_1 - c_3q_1 + c_2 - c_4q_2\right)\lambda$$
$$+ q_1^6 - c_4q_1^3 - q_2^3 - p_2^2 + c_3q_1^2 + 6q_1^2q_2^2 + c_1 - c_2q_1 - 5q_1^4q_2 - c_3q_2 + 2c_4q_1q_2,$$

and the auxiliary matrices that are given by

$$\mathbb{U}_1 = \begin{pmatrix} 0 & 1 \\ \lambda^4 - 2q_1\lambda^3 + \left(3q_1^2 - 2q_2\right)\lambda^2 + \left(-4q_1^3 + 6q_2q_1 + c_4\right)\lambda + 5q_1^4 + 3q_2^2 + c_3 - 2c_4q_1 - 12q_1^2q_2 & 0 \end{pmatrix},$$

$$\mathbb{U}_2 = \begin{pmatrix} -p_2 & \lambda + q_1 \\ (\mathbb{U}_2)_{21} & p_2 \end{pmatrix},$$

where

$$(\mathbb{U}_2)_{21} = \lambda^5 - q_1\lambda^4 + \left(q_1^2 - 2q_2\right)\lambda^3 + \left(-q_1^3 + 4q_2q_1 + c_4\right)\lambda^2 + \left(q_1^4 - 6q_2q_1^2 - c_4q_1 + 3q_2^2 + c_3\right)\lambda$$
$$- q_1^5 + c_4q_1^2 - 9q_1q_2^2 + c_2 - c_3q_1 + 8q_1^3q_2 - 2c_4q_2.$$

Transformation between jet and canonical coordinates

Finally, the map (5.4) between the jet variables and the (extended by Casimir variables c_i) Viète coordinates on the stationary manifold $\mathcal{M}_2$ is given by

$$q_1 = \frac{1}{2}r, \qquad q_2 = \frac{3}{8}r^2 + \frac{1}{2}w, \qquad p_1 = \frac{1}{4}rr_x + \frac{1}{4}w_x, \qquad p_2 = \frac{1}{4}r_x, \tag{7.33a}$$

together with

$$\begin{aligned} c_1 = h_2 &\equiv -\frac{9}{64}wr^4 - \frac{3}{8}vr^3 - \frac{3}{8}w^2r^2 - ur^2 - \frac{1}{2}vwr - \frac{1}{2}r_{2x}r - \frac{1}{4}w^3 - \frac{5}{16}r_x^2 - uw - \frac{1}{4}w_{2x}, \\ c_2 = h_3 &\equiv -\frac{9}{64}r^5 - \frac{3}{4}wr^3 - vr^2 - \frac{3}{4}w^2r - ur - vw - \frac{1}{4}r_{2x}, \\ c_3 = h_4 &\equiv -\frac{15}{64}r^4 - \frac{9}{8}wr^2 - vr - \frac{3}{4}w^2 - u, \qquad c_4 = h_5 \equiv -\frac{5}{8}r^3 - \frac{3}{2}wr - v. \end{aligned} \tag{7.33b}$$

Notice that eliminating the fields u and v from (7.33b) one obtains the relations

$$\begin{aligned} c_1 &= -\frac{5}{8}c_4r^3 + c_3r^2 - \frac{1}{2}c_4rw + c_3w - \frac{5}{32}r^6 - \frac{1}{32}r^4w + \frac{3}{4}r^2w^2 - \frac{1}{2}rr_{2x} - \frac{5}{16}r_x^2 + \frac{1}{2}w^3 - \frac{1}{4}w_{2x}, \\ c_2 &= c_3r + c_4w + \frac{3}{32}r^5 + r^3w + \frac{3}{2}rw^2 - \frac{1}{4}r_{2x}. \end{aligned} \tag{7.34}$$

Applying (7.33a) to the above Hamiltonian equations we obtain respective vector fields (7.31) together with the relations (7.34).

The inverse of the whole map (7.33) is given by

$$w = 2q_2 - 3q_1^2, \qquad w_x = 4p_1 - 8p_2q_1, \qquad r = 2q_1, \qquad r_x = 4p_2,$$

together with

$$\begin{aligned} u &= 2c_4q_1 - c_3 - 5q_1^4 + 12q_2q_1^2 - 3q_2^2, \qquad v = -c_4 + 4q_1^3 - 6q_2q_1, \\ w_{2x} &= 40c_4q_1^3 - 28c_3q_1^2 + 16c_2q_1 - 40c_4q_2q_1 + 8c_3q_2 - 4c_1 - 20p_2^2 - 76q_1^6 + 280q_2q_1^4 - 216q_2^2q_1^2 + 16q_2^3, \\ r_{2x} &= -12c_4q_1^2 + 8c_3q_1 + 8c_4q_2 - 4c_2 + 24q_1^5 - 80q_2q_1^3 + 48q_2^2q_1. \end{aligned}$$

8 Conclusions

In this article we investigated a surprising link between the stationary N-field cKdV system and a family of $N+1$ Stäckel systems. The result of this paper is that – in a very precise sense – each stationary cKdV system can be parameterized as a Stäckel system in exactly $N+1$ different ways. These different parametrizations are then shown to be equivalent through appropriate finite-dimensional analogues of Miura maps. One of the profits coming from our construction is the fact that n-time solutions of these Stäckel systems lead to a particular n-time solution of the first n systems of the N-field cKdV hierarchy.

An open problem, worth further investigation, is to find other constraints on various soliton hierarchies, including cKdV, that also lead to some classes of Stäckel systems.

Appendix

Proof of Lemma 3.2. It is immediate to see that (3.13) for $k<N$ has the form:

$$h_k = \sum_{i=0}^{k}\sum_{j=i}^{k} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \equiv f_{k+1},$$

hence (3.22a).

Notice that by (2.16) we have the equality

$$\sum_{i=0}^{N}\sum_{j=k-n}^{i+n} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) = 0 \tag{A.1}$$

valid for $k \leqslant 2n+N-1$. Taking into account that $P_{n-k+j} \neq 0$ only for $j \geqslant k-n$ and $P_{n+i-j} \neq 0$ only for $j \leqslant i+n$ one can see that if $k \geqslant N$ and $k \geqslant n$ the formula (3.13) takes the form:

$$\begin{aligned}
h_k &= \sum_{i=0}^{N}\sum_{j=\max\{i,k-n\}}^{\min\{k,i+n\}} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \equiv \left(\sum_{i=k-n+1}^{N}\sum_{j=i}^{k} + \sum_{i=0}^{k-n}\sum_{j=k-n}^{i+n}\right)\mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \\
&\overset{\text{by (A.1)}}{\equiv} \sum_{i=k-n+1}^{N}\left(\sum_{j=i}^{k} - \sum_{j=k-n}^{i+n}\right)\mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \\
&= -\sum_{i=k-n+1}^{N}\left(\sum_{j=k+1}^{i+n} + \sum_{j=k-n}^{i-1}\right)\mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \\
&\equiv -2\sum_{i=k-n+1}^{N}\sum_{j=k-n}^{i-1} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \equiv g_{k-n+1}.
\end{aligned}$$

Thus, comparing it with (3.4b), we prove the formula (3.22b) for $k \geqslant N$. The remaining case $k=N-1$ will be obtained as part of the following computation.

Similarly as above, the formula (3.13) for $n \leqslant k \leqslant N-1$ takes the form:

$$h_k = \sum_{i=0}^{k} \sum_{j=\max\{i,k-n\}}^{\min\{k,i+n\}} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \equiv \left(\sum_{i=k-n+1}^{k} \sum_{j=i}^{k} + \sum_{i=0}^{k-n} \sum_{j=k-n}^{i+n} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})$$

$$\overset{\text{by (A.1)}}{\equiv} \left(\sum_{i=k-n+1}^{k} \sum_{j=i}^{k} - \sum_{i=k-n+1}^{N} \sum_{j=k-n}^{i+n} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})$$

$$\equiv \left(\sum_{i=k-n+1}^{k} \sum_{j=i}^{k} - \sum_{i=k-n+1}^{k} \sum_{j=k-n}^{i+n} - \sum_{i=k+1}^{N} \sum_{j=k-n}^{i+n} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})$$

$$= -\left(\sum_{i=k-n+1}^{k} \sum_{j=k+1}^{i+n} + \sum_{i=k-n+1}^{k} \sum_{j=k-n}^{i-1} + \sum_{i=k+1}^{N} \sum_{j=k-n}^{i+n} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})$$

$$\equiv -\left(2 \sum_{i=k-n+1}^{k} \sum_{j=k-n}^{i-1} + \sum_{i=k+1}^{N} \sum_{j=k-n}^{i+n} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}).$$

Now, comparing with (3.4b) and (3.4c) we see that

$$h_k \equiv g_{k-n+1} + \left(2 \sum_{i=k+1}^{N} \sum_{j=k-n}^{i-1} - \sum_{i=k+1}^{N} \sum_{j=k-n}^{i+n} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})$$

$$\equiv g_{k-n+1} + \left(\sum_{i=k+1}^{N} \sum_{j=k-n}^{i-1} - \sum_{i=k+1}^{N} \sum_{j=i}^{i+n} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})$$

$$\equiv g_{k-n+1} + \left(\sum_{i=k+1}^{N} \sum_{j=k-n}^{i-1} - \sum_{i=k+1}^{N} \sum_{j=k-n}^{k} \right) \mathcal{J}_i(P_{n-k+j}, P_{n+i-j})$$

$$= g_{k-n+1} + \sum_{i=k+1}^{N} \sum_{j=k+1}^{i-1} \mathcal{J}_i(P_{n-k+j}, P_{n+i-j}) \equiv g_{k-n+1} + \tilde{g}_{k+2}.$$

Hence, we obtain (3.22c) and the remaining case of (3.22b), since the sum in the last line of the above computation vanishes for $k = N-1$.

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