Linköping Studies in Science and Technology
Licentiate Thesis No. 2025

Propagation of Opinions on Signed Graphs

Stability and Collective Wisdom

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Propagation of Opinions on Signed Graphs: Stability and Collective Wisdom

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To my Family!

Abstract

This thesis investigates how collective opinions evolve in social networks, focusing on two crucial, real-world factors: individual stubbornness and antagonistic interactions. Classical models of opinion dynamics typically assume a cooperative environment, where all individuals collaborate to reach an agreement, causing opinions to naturally draw closer to each other. However, many real-world scenarios—from political debates to online discussions—are defined by stubbornness, where individuals resist changing their stance, and by rivalry or distrust, which can actively push opinions apart.

We address these phenomena through the framework of signed networks, which explicitly encodes antagonistic relationships. Unlike models that rely on structural balance—a clean division into two opposing camps—we adopt a repelling interaction mechanism, treating negative ties as additive repulsive forces. This approach offers a more flexible and realistic representation of complex, unbalanced social structures.

The first contribution is a thorough analysis of the signed Friedkin–Johnsen (SFJ) model, which combines stubborn attachment to initial opinions with antagonistic interactions. This combination fundamentally changes system behavior: while the classical FJ model is always stable, the SFJ model can diverge. Therefore, we establish sufficient stability conditions and show that even when the model is stable, antagonism allows opinions to escape the the convex hull of the initial opinions, meaning that agents can adopt positions more extreme than anyone held at the start.

The thesis then examines repeated discussions through a concatenated SFJ model, reflecting scenarios such as a series of negotiations. Here, stability in an individual discussion does not imply stability for the repeated process. Two distinct behaviors emerge when the individual discussion rounds are stable: opinions may temporarily move apart before reconverging to consensus inside the convex hull, or they may drift farther apart with each round, leading to divergence. The first case corresponds to a transient amplification, which is a known consequence of non-normal matrices and is often referred to as reactivity. Given these possibilities, we provide sufficient stability conditions for the concatenated SFJ model. This analysis is extended to explore the dynamics in more complex scenarios, including time-varying interaction structures and bipartite (two-camp) networks.

Next, we address multidimensional opinion dynamics models, where agents debate several interdependent topics (for example, climate policy is inherently linked to economic policy). We show that a set of topics that would be stable if discussed independently can become unstable and diverge when discussed together. The analysis of repeated (concatenated) multidimensional interactions reveal diverse outcomes, from full or partial or bipartite consensus to divergence.

Finally, we shift the focus from opinion convergence to the quality of resulting collective judgement and study the wisdom of crowds problem, i.e., analyze the conditions under which a group's aggregated opinion becomes more accurate due

to collective interactions. The improvement of wisdom depends entirely on the allocation of the social power vector—the centrality measure that each individual has in the group. In classical cooperative systems, the social power vector is positive, which confines the concentration region (the set of allocations that improve wisdom) to a simplex. In contrast, signed networks allow for negative social powers, which expands the concentration region beyond the simplex, to a hyperplane. The analysis on signed networks also leads to a critical insight: agreement does not imply accuracy. The group can become confidently wrong—converging with high certainty to a false truth. We also analyze the case where the agents' initial opinions are correlated, and characterize the new concentration regions and their properties under these conditions.

Populärvetenskaplig sammanfattning

Tänk dig en livlig diskussion runt ett middagsbord. Människor delar sina åsikter, påverkar varandra och ändrar ibland uppfattning. Denna process—hur åsikter utvecklas när människor interagerar—kallas åsiktsdynamik. Den hjälper oss att förstå varför vissa grupper når enighet medan andra delas i två, eller flera, läger. De flesta traditionella modeller antar att människor påverkar varandra positivt, så att åsikterna närmar sig varandra. Men i verkligheten kan människor också ha motsatta åsikter i en fråga. För att fånga hela bilden behöver vi modeller som inkluderar både samarbete och konflikt.

En sådan modell introduceras i den första delen av min forskning. Den kombinerar två krafter: negativa relationer, där människor reagerar mot varandra, och den naturliga tendensen att hålla fast vid sin ursprungliga åsikt. Vad händer då? Åsikterna närmar sig inte alltid varandra. Istället för att närma sig varandra kan åsikterna glida isär och hamna utanför det ursprungliga intervallet, vilket gör ståndpunkterna mer extrema. Detta hjälper oss att förstå varför polarisering kan växa även när människor fortsätter att prata. Och när diskussionerna sker i flera omgångar ser vi ett intressant fenomen: Ibland stannar åsikterna aldrig utan glider längre och längre isär, och ibland hittar de fram till en överenskommelse. En sådan överenskommelse, efter en initial splittring, är vanlig eftersom motsatta individer har gemensamma vänner som fungerar som broar.

I den andra delen av min forskning undersöker vi vad som händer när människor diskuterar flera frågor samtidigt, och dessa frågor hänger ihop. Tänk på någon som älskar både matematik och fysik, men inte är särskilt förtjust i litteratur. Dessa kopplingar spelar roll. Vi har byggt en modell som fångar dessa samband och fann att de kan skapa oväntade mönster. Ibland når alla frågor en gemensam kompromiss. Ibland uppstår bara delvisa överenskommelser. Ibland når vi aldrig en gemensam ståndpunkt. Om banden mellan frågorna är för starka kan upprepade diskussioner faktiskt driva åsikterna längre ifrån varandra. Detta förklarar varför verkliga samhällsdebatter—där frågor är sammanflätade—kan vara så oförutsägbara.

Slutligen undersöks den välkända idén om "wisdom of crowds": tron att en grupps genomsnittliga åsikt ofta är mer träffsäker än någon enskild individs gissning. Vi studerade hur denna visdom förändras när människor diskuterar och påverkar varandra; blir gruppen smartare—eller mindre träffsäker? Svaret är: båda kan hända. I vissa fall blir gruppen klokare tillsammans, och i andra fall blir den mer självsäker men rör sig bort från sanningen. Vi undersökte också vad som händer när människors initiala åsikter inte är oberoende utan hänger ihop, vilket ofta är fallet i verkligheten. Dessa beroenden förändrar villkoren för när modellerna förbättrar visdomen.

Tillsammans visar dessa studier att vägen till enighet är långt ifrån enkel. Konflikter, viljan att hålla fast vid ursprungliga åsikter och kopplingar mellan olika frågor kan leda antingen till enighet eller till oenighet. Och även om grupper ibland blir klokare tillsammans, kan de lika gärna bli mer självsäkra—och mer fel ute. Att

förstå dessa dynamiker är avgörande för att kunna tolka polarisering, samhällsdebatter och beslutsfattande i en alltmer sammanlänkad värld.

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Linköping, January 2026
Muhammad Ahsan Razaq

Contents

Part I

Background

1

Introduction

Multiagent systems offer a powerful framework for understanding phenomena in nature and society [1–3]. These systems consist of multiple interconnected components—often referred to as agents or nodes—that interact through a network of edges or links. Such systems exhibit behaviors that cannot be understood by analyzing individual components in isolation, as their collective dynamics emerge from local interactions. Examples include social networks intended broadly as networks where individuals exchange information, with applications in political systems [4–6], financial markets [7–9], and epidemiological systems [10–12]. Understanding such systems helps explain why they can remain stable under disturbances, adapt to changing conditions, and display unexpected collective behaviors that arise from simple local interactions.

Opinion dynamics is an example of a multiagent system, concerned with how individual opinions, beliefs, or attitudes evolve through interpersonal interactions and how these local exchanges shape collective outcomes at the group or societal level [13–18]. Depending on the structure and strength of interpersonal influences, the resulting patterns can include consensus, polarization, clustering, or persistent disagreement [15, 16]. The mechanisms driving opinion dynamics can be represented through two main mathematical frameworks: stochastic models and deterministic models. Stochastic models capture probabilistic updates, such as Bayesian belief model [19]. Deterministic models, on the other hand, describe opinion evolution under continuous-time (CT) or discrete-time (DT) dynamics, either linear or nonlinear, allowing for rigorous analysis of stability and convergence [14, 15, 18]. Prominent examples of deterministic opinion dynamics models include the linear DeGroot model [20], which captures opinion averaging through repeated weighted interactions among agents, and the affine Friedkin–Johnsen (FJ) model [21], which extends DeGroot's framework by introducing individual stubbornness,

allowing agents to partially retain their initial opinions. A widely studied nonlinear extension is the Hegselmann–Krause bounded-confidence model [22], where agents only interact with neighbors whose opinions lie within a confidence bound, leading to phenomena such as opinion clustering and fragmentation. Among deterministic approaches, linear models have become foundational due to their simplicity and analytical tractability, capturing essential features of social influence while enabling the use of tools from linear algebra, spectral theory, and control [15].

To describe a collective decision-making process in this context, two components are essential: a representation of how individuals hold and exchange opinions, and a dynamical model describing how these opinions evolve over time. A common approach is to use a state-space representation, where each agent's opinion is treated as a state, and interactions are encoded through a weighted directed graph. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote the interaction graph with n nodes (agents). The adjacency matrix of the graph $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ collects the influence weights, with $a_{ij} \geq 0$ representing the influence of agent j on agent i. The degree of node i is $\xi_i = \sum_j a_{ij}$, and the degree matrix is $\Sigma = \operatorname{diag}(\xi_1, \ldots, \xi_n)$. The Laplacian is then $L = \Sigma - A$, a key operator governing the dynamics.

In CT, the opinion vector $\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^\top$ evolves according to

$$\dot{\mathbf{x}}(t) = f\big(\mathbf{x}(s) : s \in [0, t]\big), \tag{1.1}$$

where $f : \mathcal{C}([0, t], \mathbb{R}^n) \to \mathbb{R}^n$ is a functional that determines the current rate of change based on the history of opinions over the interval $[0, t]$ and the underlying network structure.

A fundamental linear time invariant case is when $f(\mathbf{x}) = -L\mathbf{x}$, where L is the Laplacian of the interaction graph $\mathcal{G}$. This case corresponds to the CT DeGroot model, which is a diffusion process: the rate of change of each opinion depends on the differences with its neighbors. When the graph is connected—meaning that every agent can influence, directly or indirectly, every other agent in the network—this simple model steers all opinions to a common value, showing how consensus can form without any central authority [15, 20].

In DT, the opinion dynamics can be expressed in general form

$$\mathbf{x}(k + 1) = g\big(\mathbf{x}(k), \mathbf{x}(k - 1), \ldots, \mathbf{x}(0)\big), \tag{1.2}$$

where $g : (\mathbb{R}^n)^{k+1} \to \mathbb{R}^n$ depends on the update mechanism and the entire history of opinions. A common special case is a memoryless update rule with $g(\mathbf{x}) = W\mathbf{x}$, where $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ is a row-stochastic matrix encoding normalized influence weights. Each entry $w_{ij} \geq 0$ represents the weight that agent i assigns to agent j's opinion. This formulation corresponds to the DT DeGroot model, which captures repeated interpersonal averaging: at each interaction step, agents update their opinions by taking a weighted average of their neighbors' opinions. Under the assumption of a connected graph, both the CT and DT DeGroot models converge to consensus:

$$\lim_{t\to\infty} \mathbf{x}(t) = \lim_{k\to\infty} \mathbf{x}(k) = \mathbb{1}\big(\mathbf{w}^\top \mathbf{x}(0)\big), \tag{1.3}$$

where $\mathbf{w}$, often referred to as the social power vector, reflects the centrality of each agent. So, the consensus value is a weighted average of initial opinions determined by the network structure.

The FJ model extends the DeGroot dynamics by incorporating stubbornness, represented by a diagonal matrix $\Theta = \text{diag}(\theta_i)$ with $0 \leq \theta_i \leq 1$ [21]. The FJ model modifies the dynamics in (1.1) and (1.2) as follows:

$$\text{CT: } f(\mathbf{x}, \mathbf{x}(0)) = -(I - \Theta)L\mathbf{x} + \Theta\mathbf{x}(0), \qquad \text{DT: } g(\mathbf{x}, \mathbf{x}(0)) = (I - \Theta)W\mathbf{x} + \Theta\mathbf{x}(0),$$

which can be interpreted as a convex combination of the DeGroot update and the initial opinions, i.e., both mappings depend on the current opinion profile and the initial opinions. The parameter θ_i quantifies the degree of stubbornness of agent i: $\theta_i = 1$ denotes a fully stubborn agent, who completely disregards the opinions of the others and retains its initial opinion throughout the process, while $\theta_i = 0$ corresponds to no stubbornness, meaning the agent is entirely open to influence and updates its opinion solely based on its neighbors. When $\Theta = \text{diag}(\mathbf{0})$, the model reduces to the DeGroot model, and for $\Theta \neq \text{diag}(\mathbf{0})$, the system becomes affine rather than linear. Under mild conditions (e.g., a connected graph), it converges to a unique equilibrium that does not coincide with consensus. Instead, the limiting opinions strictly lies inside the convex hull of the initial conditions, representing a configuration where agents move closer to each other compared to their initial positions, yet retain persistent differences due to stubbornness.

Classical opinion dynamics models, including DeGroot and FJ, assume cooperative interactions, i.e., all influence weights are nonnegative. In many social settings, however, relationships can be antagonistic—reflecting rivalry, distrust, or ideological opposition—and thus tend to push opinions apart rather than bring them closer. To capture such effects, the framework is extended to signed networks, where the entries of the adjacency matrix A (CT) or interaction matrix[1] W (DT) may be positive (supportive) or negative (antagonistic).

Introducing antagonistic ties fundamentally changes the behavior of opinion dynamics compared to purely cooperative settings [23–26]. A central concept in this context is the notion of structural balance, which determines whether the network can be partitioned into two internally friendly groups with antagonistic relations across them. When this condition holds, opinion dynamics often lead to predictable outcomes such as bipartite consensus, where two opposing camps emerge. Yet most real-world networks—ranging from biological systems [27], social networks [5], and even parliamentary systems [6]—are structurally unbalanced. This imbalance makes convergence more complex, often resulting in polarization, oscillations, or persistent disagreement.

Based on this, two main formulations have been proposed for defining Laplacians (and their DT counterparts) on signed graphs [24, 25]. The first formulation is the opposing Laplacian, where a negative edge (j, i) reverses the sign of agent j's opinion in agent i's update rule and $\xi_i = \sum_j |a_{ij}|$ in the degree matrix Σ. In this

[1] In this thesis, both terms interaction matrix and influence matrix are used to refer to W.

case, stability is guaranteed, and the convergence outcome depends on the structural balance of the graph. Under this condition, the signed DeGroot model with the opposing Laplacian converges to bipartite consensus, where opinions polarize into two opposing camps; otherwise, all opinions collapse to zero [15, 23].

The second formulation is the repelling Laplacian [24], where negative edges contribute additively and $\xi_i = \sum_j a_{ij}$, creating a repulsive effect that does not require structural balance. In this formulation of the DeGroot model, the long-term behavior depends on the spectral properties of the signed Laplacian (or its DT counterpart). When the system is stable, opinions do not necessarily converge to a single consensus value; instead, they may approach a configuration determined by the network structure. If the Laplacian has multiple eigenvalues equal to zero (or, in the DT case, the interaction matrix has multiple eigenvalues at one), the system is only marginally stable: opinions remain bounded but may fail to converge, leading to clustering within groups or persistent disagreement [25, 28]. Conversely, if the spectral conditions indicate instability, trajectories diverge.

The signed FJ (SFJ) model with the opposing Laplacian has been studied in the literature [29, 30], where opinions converge asymptotically toward a bipartite configuration when the network is structurally balanced, and toward zero otherwise. In Paper A, the SFJ model is extended to the repelling formulation, where stability conditions are formally derived.

However, a fundamental characteristic of the FJ framework is that inherent stubbornness prevents the system from reaching a global consensus in a single discussion, leading instead to a state of persistent disagreement. This dynamics mirrors real-world decision-making, where complex negotiations often unfold over a sequence of repeated discussions rather than concluding in a single meeting. To capture this iterative process, a framework of concatenated interactions with two time scale has been proposed [31, 32].

In this setting, the opinions $\mathbf{x}(s,t)$ evolve on a "fast" time scale t during the s-th discussion round according to the FJ model. Due to the stability of the interaction matrix, these opinions converge exponentially fast to a unique equilibrium point $\mathbf{x}(s,\infty)$. The concatenation is then modeled by linking consecutive FJ models along the "slow" time scale s, such that the initial opinions for the next discussion event are set equal to the final opinions of the previous one:

$$\mathbf{x}(s+1,0) = \mathbf{x}(s,\infty).$$

This concatenation yields a DT evolution of opinions across meetings, governed by the equation

$$\mathbf{x}(s+1,0) = P\mathbf{x}(s,0),$$

where $P = (I-(I-\Theta)W)^{-1}\Theta$ is a matrix capturing the combined effect of network influence and stubbornness. In the standard FJ dynamics, the equilibrium opinions are constrained to lie strictly within the convex hull of the initial conditions. Consequently, over a sequence of concatenated discussions, this repeated contraction of the opinion space drives the system asymptotically toward a global

consensus, provided the interaction graph is strongly connected [32, 33]. Notably, the social power vector governing this consensus depends on both the interaction graph and the stubbornness values of the individuals. Extending this framework to antagonistic settings, Paper A explores the concatenated SFJ model, establishing spectral stability conditions on the interaction matrix W sufficient for convergence, and extending the analysis to bipartite networks and interaction structures that vary across discussion rounds.

In realistic social settings, individuals rarely discuss a single topic in isolation; rather, they form opinions on multiple interdependent subjects simultaneously. To capture this, a multidimensional framework has been developed in which dynamics are governed jointly by the social interaction matrix and by a topic-dependency matrix [34–36]. This coupling models cognitive consistency—such as aligning views on general dietary preferences with specific food choices—creating a belief system where the final equilibrium is constrained by the interplay of peer pressure and internal logic [35]. In the cooperative case, despite this coupling, the DeGroot dynamics steer opinions toward consensus, while the FJ dynamics retain the contractive nature. Consequently, in the concatenated framework, this ensures asymptotic convergence to a global consensus, provided the network is connected.

While this framework has been extended to signed networks, previous studies have primarily restricted their analysis to the opposing Laplacian formulation [37, 38]. Due to specific structural properties that ensure diagonal dominance, these opposing models remain inherently stable regardless of the network's sign pattern. Whether in the signed DeGroot model or the concatenated SFJ model, the system typically converges to a bipartite consensus, polarizing either across agents (forming hostile camps) or across topics (holding opposing views on linked issues), depending on whether the structural balance characterizes the social network W or the topic dependency graph C.

Addressing the more complex dynamics of the repelling formulation, Paper B investigates the multidimensional SFJ model and its concatenated extension. In this work, we find stability conditions for the collective dynamics, accounting for the joint spectral properties of the social interaction and topic-dependency matrices.

Opinion dynamics is concerned not only with whether opinions converge but also with the accuracy of the resulting collective judgment. The concept of "wisdom of crowds" formalizes this idea, referring to the phenomenon where the aggregated opinion of a group is more accurate than that of most individuals, typically measured by the proximity of the limiting opinion profile to an underlying truth [39, 40].

To quantify this, we assume that the initial opinion $x_i(0)$ of each agent is a random variable drawn from a distribution with a common mean ζ (representing the underlying truth) but with distinct variances σ_i^2. Here, the variance determines the expertise of an agent: a lower variance implies that the agent is an expert with higher precision and reliability, while a larger variance indicates a naive agent with greater uncertainty. A model is said to improve collective wisdom if and only if

it satisfies two conditions. First, it must be mean accurate, meaning that the expected value of the group's final aggregate opinion remains aligned with the truth ζ. Second, it must exhibit variance concentration, meaning that the variance of the final weighted estimate is strictly lower than that of the initial simple average:

$$\sum_{i=1}^{n} w_i^2 \sigma_i^2 < \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2,$$

where w_i is the social power of agent i in the network.

In classical cooperative networks, the dynamics are typically mean accurate by construction; consequently, the improvement of wisdom depends entirely on whether the resulting social power allocation $\mathbf{w}$ achieves variance concentration. Since $\mathbf{w}$ is dictated by the network topology, structural features play a critical role: highly connected and balanced networks tend to promote wisdom, whereas echo chambers often undermine it [40, 41]. Recent studies have formalized this relationship geometrically, establishing that collective accuracy is improved if and only if the social power vector lies within a specific convex concentration region [42]. Paper C generalizes this framework to signed networks, demonstrating how antagonistic interactions and negative social powers fundamentally alter the geometry of these regions and the conditions for collective accuracy, particularly when initial opinions are correlated.

1.1 Contributions

This thesis investigates opinion dynamics on signed social networks, with emphasis on how antagonistic interactions and persistent initial opinions influence collective outcomes. Papers A and B focus on the SFJ model and its extensions. Paper A introduces the SFJ model under the repelling Laplacian formulation and explores its concatenated version, analyzing stability conditions and revealing transient amplification effects where opinions temporarily diverge before reconverging. Paper B generalizes this framework to multiple interdependent topics, showing that topic coupling can destabilize otherwise stable dynamics and lead to diverse behaviors such as full consensus, partial consensus, or divergence. Paper C shifts the perspective to collective accuracy, extending the wisdom of crowds framework to signed networks and characterizing how antagonistic ties reshape variance concentration and the geometry of social power allocations, uncovering scenarios where higher confidence in agreement does not imply improved accuracy, meaning that opinions become more concentrated yet may collectively drift away from the true value.

The main contributions of the thesis are detailed below.

- Formulation of the SFJ model under repelling interactions in both DT and CT, accompanied by a rigorous stability analysis. The study shows that opinions may converge to an equilibrium point outside the convex hull of the initial opinions. (Paper A)
- Stability analysis of concatenated SFJ dynamics, showing that repeated dis-

cussions can lead to consensus or divergence even when individual rounds are stable, and documenting transient amplification effects where opinions temporarily leave the convex hull before reconverging. The analysis is further extended to bipartite networks and time-varying interaction structures. (Paper A)

- Development of a multidimensional SFJ framework that incorporates interdependent topics. Sufficient stability conditions are provided for the concatenated case, and it is shown that topic coupling can destabilize otherwise stable systems, producing behaviors such as full consensus, bipartition, or divergence. (Paper B)
- Extension of the wisdom of crowds framework to signed networks, including DeGroot, SFJ, and concatenated SFJ models. Concentration regions for social power allocations are characterized, showing a shift from a simplex (unsigned case) to a hyperplane (signed case) allowing negative weights. Scenarios are identified where higher confidence in the decision does not imply improved accuracy. (Paper C)
- Incorporation of correlated initial opinions into the wisdom of crowds analysis demonstrates how correlation reshapes concentration regions and influences the feasibility of variance reduction under signed interactions. The study reveals that, for certain correlation structures, the social power vector that maximizes wisdom may lie outside the simplex, necessitating negative social powers (signed networks). Moreover, in some cases, the collective wisdom cannot surpass the best individual's wisdom. (Paper C)

1.2 Thesis Outline

Part I: Background

Part I introduces the theoretical material used throughout the thesis. The preliminary material includes concepts and notions of graph and matrix theory.

Part II: Publications

Part II of this thesis contains three edited research papers listed below.

Paper A: Signed Friedkin–Johnsen models: opinion dynamics with stubbornness and antagonism

Edited version of the paper:

> Muhammad Ahsan Razaq and Claudio Altafini. Signed Friedkin-Johnsen models: Opinion dynamics with stubbornness and antagonism. *IEEE Transactions on Automatic Control*, 70(8):5037–5051, 2025.

Summary This paper introduces the SFJ model for opinion dynamics under antagonistic interactions, analyzing both DT and CT settings. Unlike the classical FJ model, which is always stable, the SFJ model can become unstable under repelling configurations; therefore, stability conditions for a single-discussion SFJ model are first formulated. The analysis reveals that opinions may leave the convex hull of initial conditions—a phenomenon that is absent in classical models. The study then investigates concatenated SFJ dynamics (repeated discussions), identifying stability conditions and two possible behaviors when individual discussions (single SFJ model) are stable: (i) opinions repeatedly leave the convex hull and diverge, making the concatenated SFJ model unstable, and (ii) opinions that may initially leave the convex hull are later reabsorbed, leading to consensus. For the stable single-discussion case, these behaviors correspond to divergence through repeated excursions or temporary excursions followed by consensus. The analysis is extended to bipartite networks, where opinions converge to bipartite consensus instead of full consensus when the model is stable. Finally, the paper addresses time-varying interaction structures, deriving stability conditions under changes in the interaction matrix and stubbornness values. It identifies cases where consensus is guaranteed when either the interaction matrix varies with fixed stubbornness or when stubbornness changes uniformly across agents, and it provides a common Lyapunov function that ensures consensus under arbitrary variations in both parameters.

Contribution and background Muhammad Ahsan Razaq (A.R.) contributed substantially to the theoretical development, analytical proofs, implementation, numerical simulations, and manuscript preparation. The research direction and initial problem formulation were developed in close collaboration with Claudio Altafini (C.A.), who provided significant conceptual guidance, manuscript revisions, and critical feedback throughout the process.

Paper B: Multidimensional signed Friedkin–Johnsen model

Edited version of the paper:

> Muhammad Ahsan Razaq and Claudio Altafini. Multidimensional signed Friedkin-Johnsen model. In *2024 IEEE 63rd Conference on Decision and Control (CDC)*, pages 5304–5309. IEEE, 2024.

Summary This paper extends the SFJ framework to a multidimensional setting, where agents hold opinions on several interdependent topics. Social influence and topic coupling are integrated and spectral conditions for stability and convergence are derived for both DT and CT. A key insight is that stability in isolated discussions does not imply stability when topics are interdependent: concatenated interactions can either drive the system toward consensus or amplify disagreements, depending on the interplay between the social network and the topic-dependency structure—even when individual topics would converge independently. The analysis also reveals how topic coupling shapes long-term behavior, leading to outcomes

such as full consensus, bipartite consensus, partial agreement across topics, or divergence.

Contribution and background The idea of extending the SFJ framework to multidimensional opinion dynamics originated with A.R., who led the theoretical development, proofs, and implementation. C.A. provided guidance on refining the approach and offered detailed feedback and revisions to strengthen the manuscript.

Paper C: Wisdom of crowds effects under antagonistic interactions and correlated opinions

Edited version of the paper:

> Muhammad Ahsan Razaq and Claudio Altafini. Wisdom of crowds effects under antagonistic interactions and correlated opinions. *arXiv preprint arXiv:2510.19123*, 2025, under review.

Summary This paper studies the evolution of the wisdom of crowds property under opinion dynamics on signed networks, where interactions can be both cooperative and antagonistic, and where initial opinions may be correlated. The wisdom of crowds effect is quantified in terms of variance reduction: a model is said to improve the wisdom if the variance of the group's aggregate opinion decreases compared to the initial variance.

A central concept in the analysis is the concentration region, defined as the set of social power allocations (weights assigned to individuals in the asymptotic equilibrium) that lead to a variance reduction of the average opinion. For unsigned networks, this region is the subset of a standard simplex, since all social powers are nonnegative and sum to one. Our results show that introducing antagonistic ties fundamentally changes this structure: the concentration region becomes a subset of a hyperplane, allowing for negative social powers. This structural difference implies that some individuals' opinions are effectively assigned negative weights, counteracting their influence in the aggregation process for possible improvement in wisdom. Furthermore, we observe that variance may exhibit transient behavior: it can decrease initially—suggesting improved wisdom—before increasing again over repeated concatenated discussions. This highlights that short-term improvements do not guarantee long-term accuracy.

For bipartite networks, we find an interesting effect: the concentration region can favor allocations that reduce variance but lead the group towards the incorrect consensus value. In other words, the group becomes more confident while systematically moving away from the true value.

Lastly, we also extend the analysis to the case of correlated initial opinions. Correlations reshape the concentration region, altering the feasibility of variance reduction and introducing additional constraints on the social power vector. Correlated opinions also lead to several notable effects: the optimal social power vector may

lie outside the simplex, requiring signed networks to maximize wisdom; and, in some cases, wisdom cannot fall below the minimum individual variance.

Contribution and background The research topic was identified during a discussion involving A.R., C.A., and Ye Tian. A.R. proposed the extension to correlated opinions and carried out the theoretical analysis, proofs, implementation, manuscript preparation, and numerical simulations. C.A. contributed conceptual insights, critical revisions to the manuscript, and critical feedback throughout the process.

Publications

Works published by the author of this thesis during his doctoral studies are listed below in chronological order. Publications indicated by a ⋆ are included in the thesis.

Muhammad Ahsan Razaq and Claudio Altafini. Propagation of stubborn opinions on signed graphs. In *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 491–496. IEEE, 2023.

⋆Muhammad Ahsan Razaq and Claudio Altafini. Multidimensional signed Friedkin-Johnsen model. In *2024 IEEE 63rd Conference on Decision and Control (CDC)*, pages 5304–5309. IEEE, 2024.

⋆Muhammad Ahsan Razaq and Claudio Altafini. Signed Friedkin-Johnsen models: Opinion dynamics with stubbornness and antagonism. *IEEE Transactions on Automatic Control*, 70(8):5037–5051, 2025.

Muhammad Ahsan Razaq and Claudio Altafini. Wisdom of crowds in signed opinion dynamics models. In *2025 64th IEEE Conference on Decision and Control (CDC)*. IEEE, 2025.

⋆Muhammad Ahsan Razaq and Claudio Altafini. Wisdom of crowds effects under antagonistic interactions and correlated opinions. *arXiv preprint arXiv:2510.19123*, 2025, under review.

Works published by the author of this thesis before starting his doctoral studies are listed below in chronological order.

Muhammad Ahsan Razaq, Muhammad Rehan, Choon Ki Ahn, Abdul Qayyum Khan, and Muhammad Tufail. Consensus of one-sided Lipschitz multiagents under switching topologies. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 51(3):1485–1495, 2019.

Muhammad Ahsan Razaq, Muhammad Rehan, Muhammad Tufail, and Choon Ki Ahn. Multiple Lyapunov functions approach for consensus of one-sided Lipschitz multi-agents over switching topologies and input saturation. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 67(12):3267–3271, 2020.

Muhammad Ahsan Razaq, Muhammad Rehan, Fatima Tahir, and Mohammed Chadli. H_∞ leader-based consensus of non-linear multi-agents over switching graphs and disturbances using multiple Lyapunov functions. *IET Control Theory & Applications*, 14(20):3395–3405, 2020.

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2

Matrix Theory and Perron-Frobenius Theory

The first part of the chapter introduces notation and foundational concepts from matrix theory, including spectral theory, and numerical radius, primarily based on [43] and [44]. The second part of the chapter focuses on Perron–Frobenius (PF) theory with extension of PF theory to signed matrices.

2.1 Matrix Theory

This section presents some preliminary definitions and concepts from matrix theory, focusing on the characterization of eigenvalues.

2.1.1 Notation

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. For a complex number $z \in \mathbb{C}$, its real and imaginary parts are denoted by $\Re(z)$ and $\Im(z)$. Scalars are represented by lowercase letters such as $a, b, c, \ldots$, vectors by bold lowercase letters such as $\mathbf{x}, \mathbf{y}$, and matrices by uppercase letters such as A, B, W, L. For a matrix $Q \in \mathbb{C}^{n\times n}$, Q^* denotes its conjugate transpose. Furthermore, any square matrix Q can be decomposed into its Hermitian and skew-Hermitian parts, defined as

$$H(Q) = \frac{1}{2}(Q + Q^*), \quad \text{and} \quad S(Q) = \frac{1}{2}(Q - Q^*),$$

respectively. The identity matrix is denoted by I, while $\mathbf{0}$ and $\mathbb{1}$ represent the zero vector and the all-ones vector, respectively. For a vector $\mathbf{x} \in \mathbb{R}^n$, the diagonal matrix $\operatorname{diag}(\mathbf{x})$ places the entries of $\mathbf{x}$ along its diagonal. A matrix Q is *reducible* if $\exists$ a permutation matrix P such that $P^\top QP$ is block upper-triangular; it is *irreducible* if no such P exists.

2.1.2 Eigenvalues and Eigenvectors

Let $Q \in \mathbb{R}^{n\times n}$ be a square matrix. A scalar $\lambda \in \mathbb{C}$ is called an *eigenvalue* of Q if $\exists$ a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ (resp. $\mathbf{w} \in \mathbb{C}^n$) such that $Q\mathbf{v} = \lambda\mathbf{v}$ (resp. $\mathbf{w}^\top Q = \lambda\mathbf{w}^\top$). In this case, $\mathbf{v}$ (resp. $\mathbf{w}$) is referred to as a *right (resp. left) eigenvector* associated with λ. The *spectrum* of a matrix $Q \in \mathbb{R}^{n\times n}$ is the set of its eigenvalues: $\Lambda(Q) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The eigenvalues are typically ordered by their real parts: $\Re(\lambda_1) \geq \Re(\lambda_2) \geq \cdots \geq \Re(\lambda_n)$.

Two important quantities derived from the spectrum are the *spectral radius* and the *spectral abscissa*. The spectral radius is defined as $\rho(Q) = \max_i |\lambda_i(Q)|$, and measures the largest magnitude among the eigenvalues. The spectral abscissa is given by $\alpha(Q) = \max_i \Re(\lambda_i(Q)) = \Re(\lambda_1)$, and corresponds to the largest real part among the eigenvalues. The *dominant right (resp. left) eigenvector* of Q is a right (resp. left) eigenvector associated with an eigenvalue $\lambda \in \Lambda(Q)$ such that either $|\lambda| = \rho(Q)$ or $\Re(\lambda) = \alpha(Q)$, depending on the context. In DT systems, dominance is typically defined with respect to the spectral radius, while in CT systems, it is defined with respect to the spectral abscissa.

The *kernel* (or null space) of a matrix Q is defined as $\ker(Q) = \{\mathbf{x} \in \mathbb{R}^n : Q\mathbf{x} = \mathbf{0}\}$, and its dimension is referred to as the *corank* of Q, i.e., $\operatorname{corank}(Q) = \dim(\ker(Q))$. The *range* (or column space) of Q is the set $\operatorname{range}(Q) = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} = Q\mathbf{x}\}$, and its dimension is called the *rank* of Q. The rank satisfies $\operatorname{rank}(Q) = \operatorname{rank}(Q^\top) = n - \operatorname{corank}(Q)$.

2.1.3 Numerical Range

The *numerical range* (or field of values) of a matrix $Q \in \mathbb{C}^{n\times n}$, denoted $\omega(Q)$, is defined as

$$\omega(Q) = \{\mathbf{x}^* Q\mathbf{x} : \mathbf{x} \in \mathbb{C}^n, \|\mathbf{x}\| = 1\},$$

representing all Rayleigh quotients over the unit sphere. By the Toeplitz–Hausdorff theorem [45], $\omega(Q)$ is a compact, convex subset of $\mathbb{C}$ that always contains the spectrum, i.e., $\Lambda(Q) \subset \omega(Q)$. The *angular numerical range* (or angular field of values) of Q, denoted $\omega'(Q)$, is given by

$$\omega'(Q) = \{\mathbf{x}^* Q\mathbf{x} : \mathbf{x} \in \mathbb{C}^n \setminus \{0\}\}.$$

The boundary of the numerical range $\omega(Q)$ is defined as $\partial\omega(Q)$. Two key quantities associated with the numerical range are the *numerical radius* and the *numerical abscissa*. The numerical radius is defined as

$$r(Q) = \max\{|z| : z \in \omega(Q)\},$$

and satisfies the bounds

$$\rho(Q) \leq r(Q) \leq \|Q\|_2 \leq 2r(Q),$$

where $\|\cdot\|_2$ is the spectral norm. The numerical radius serves as a norm equivalent to the spectral norm and provides an upper bound on the growth of trajectories in linear systems, and $r(Q) < 1$ implies Schur stability in DT systems.

The numerical abscissa is defined as

$$\mu(Q) = \max\{\Re(z) : z \in \omega(Q)\},$$

and plays a crucial role in stability analysis: $\mu(Q) < 0$ implies Hurwitz stability in CT systems. Furthermore, the numerical abscissa bounds the spectral abscissa: $\alpha(Q) \leq \mu(Q)$.

A point $\zeta \in \partial\omega(Q)$ is called a *sharp point* if it lies on the boundary of the numerical range and forms a non-smooth corner. Formally, ζ is a sharp point of $\omega(Q)$ if there are angles ϑ_1 and ϑ_2 with $0 \leq \vartheta_1 < \vartheta_2 < 2\boldsymbol{\pi}$ such that, for all $\vartheta \in (\vartheta_1, \vartheta_2)$, the following holds:

$$\Re(e^{i\vartheta}\zeta) = \max\{\Re(\beta) : \beta \in \omega(e^{i\vartheta}Q)\}.$$

A sharp point $\zeta \in \partial\omega(Q)$ is an eigenvalue of Q. For a matrix $Q \in \mathbb{R}^{n\times n}$, if $\rho(H(Q)) = \rho(Q)$ is simple and strictly dominant (resp., $\mu(Q) = \mu(H(Q))$ is simple), it implies the existence of a sharp point in $\omega(Q)$ at $\rho(Q)$ (resp., $\mu(Q)$). This leads to the following Lemma about eigenvectors related to sharp points.

Lemma 2.1 (Lemma 3.10 of [46]). *For a matrix $Q \in \mathbb{R}^{n\times n}$ and $\zeta \in \mathbb{R}$, $\zeta \in \omega(Q)$ is a sharp point iff the following two conditions are satisfied:*

- *$\zeta \in \Lambda(Q)$ and $\zeta \in \partial\omega(Q)$.*
- *Q and $H(Q)$ have a common eigenspace corresponding to the eigenvalue ζ.*

The lemma above implies that the left and right eigenvectors associated with the eigenvalue $\zeta \in \mathbb{R}$ of a matrix Q are the same if $\zeta \in \omega(Q)$ is a sharp point. Also, for sharp points, we can have a notion of strict dominance.

A *strictly dominant sharp point* is a sharp point whose modulus is strictly larger than that of any other boundary point $\partial\omega(Q)$ under all rotations, i.e., $\zeta \in \omega(Q)$ is a sharp point and $|\zeta| > |\varphi|\ \forall\, \varphi \in \partial\omega(Q)$, $\varphi \neq \zeta$.

Properties of Numerical Range. Several invariance properties hold for the numerical range and have implications for eigenvalues. For any scalar $\alpha \in \mathbb{C}$, adding a multiple of the identity shifts the numerical range by α, i.e., $\omega(Q+\alpha I) = \omega(Q) + \alpha$. Similarly, scaling the matrix by a scalar β scales the numerical range by the same factor: $\omega(\beta Q) = \beta\,\omega(Q)$.

The numerical range also satisfies subadditivity: for any two matrices of the same size, $\omega(Q + R) \subset \omega(Q) + \omega(R)$, where the sum denotes the Minkowski sum of sets, i.e., the set of all point-wise additions of an element from each set. If R is positive semi-definite, then $\omega(QR) \subseteq \omega(Q)\omega(R)$, and if $0 \notin \omega(R)$, then $\omega(QR^{-1}) \subseteq \omega(Q)/\omega(R)$, with element-wise interpretation.

For two matrices $Q \in \mathbb{C}^{m\times m}$ and $R \in \mathbb{C}^{n\times n}$, the block diagonal matrix of Q and

R, denoted by $Q \oplus R$, is defined as

$$Q \oplus R = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}.$$

The numerical range of $Q \oplus R$ satisfies $\omega(Q \oplus R) = \mathrm{Co}(\omega(Q) \cup \omega(R))$, where $\mathrm{Co}(\,\cdot\,)$ denotes the convex hull. For a matrix $Q \in \mathbb{C}^{n\times n}$ and an index set $J \subseteq \{1, \ldots, n\}$, let $Q(J)$ denote the principal submatrix obtained by selecting rows and columns indexed by J. Then the numerical range satisfies the submatrix inclusion property: $\omega(Q(J)) \subseteq \omega(Q)$. This inclusion shows that restricting Q to a principal submatrix cannot enlarge its numerical range.

Isometric Projections. If $U \in \mathbb{C}^{n\times k}$ satisfies $U^*U = I_k$, the transformation U^*QU represents an *isometric projection*. This operation reduces the numerical range while preserving convexity, specifically satisfying the inclusion $\omega(U^*QU) \subseteq \omega(Q)$. The equality $\omega(U^*QU) = \omega(Q)$ holds when $k = n$, a case known as *unitary similarity invariance*.

Congruent Matrices. A more general transformation is that of congruence. Two matrices $Q, R \in \mathbb{C}^{n\times n}$ are said to be *congruent* if there exists a nonsingular matrix P such that $R = P^*QP$. Unlike isometric projections, general congruence does not preserve the standard numerical range; however, it ensures the invariance of the *angular numerical range*, i.e., $\omega'(Q) = \omega'(R)$. These properties are particularly relevant for quadratic forms and inertia analysis. This connection is formalized by Sylvester's Law of Inertia [43, Theorem 4.5.8]: if two Hermitian matrices Q and R are congruent, they share the same inertia, meaning they possess the same number of positive, negative, and zero eigenvalues.

Similar Matrices. Two matrices $Q, R \in \mathbb{C}^{n\times n}$ are said to be *similar* if $\exists$ a nonsingular matrix P such that $R = P^{-1}QP$. Similarity is an equivalence relation that preserves the characteristic polynomial, and hence similar matrices have the same spectrum, i.e., $\Lambda(Q) = \Lambda(R)$. However, similarity does not preserve the numerical range. In fact, $\omega(Q)$ and $\omega(R)$ can differ significantly $\omega(Q) \neq \omega(R)$ because the numerical range depends on the Euclidean inner product.

Definiteness of a matrix. A matrix $Q \in \mathbb{C}^{n\times n}$ is called *positive definite*[1], denoted $Q \succ 0$, if its associated quadratic form satisfies $\mathbf{x}^*Q\mathbf{x} > 0$, $\forall \mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. If Q is positive definite, then all its eigenvalues have strictly positive real parts, and the numerical range $\omega(Q)$ lies entirely in the open right half-plane of the complex plane. Similarly, Q is *positive semi-definite*, denoted $Q \succeq 0$, if $\mathbf{x}^*Q\mathbf{x} \geq 0$, $\forall \mathbf{x} \in \mathbb{C}^n$, in which case all eigenvalues have nonnegative real parts, and $\omega(Q)$ lies in the closed right half-plane, including the origin.

Analogously, Q is *negative definite* ($Q \prec 0$) if $\mathbf{x}^*Q\mathbf{x} < 0$ for all nonzero $\mathbf{x}$, and *negative semi-definite* ($Q \preceq 0$) if $\mathbf{x}^*Q\mathbf{x} \leq 0$ for all $\mathbf{x}$. For these cases, the eigenval-

[1] In this thesis, definiteness is defined via the quadratic form and does not require Q to be symmetric. Symmetric matrices imply real eigenvalues.

ues have strictly negative or nonpositive real parts, respectively, and the numerical range lies entirely in the open or closed left half-plane.

Radial Matrices. A matrix $Q \in \mathbb{C}^{n\times n}$ is called *radial* if its spectral radius, spectral norm, and numerical radius coincide, i.e., $\rho(Q) = \|Q\|_2 = r(Q)$. Equivalently, Q is radial *iff* it satisfies $\rho(Q)I - Q^*Q \succeq 0$. Radial matrices exhibit strong geometric regularity: their numerical range equals the convex hull of their spectrum, $\omega(Q) = \mathrm{Co}(\Lambda(Q))$. For general matrices, the difference between $\rho(Q)$ and $r(Q)$ represents the degree of non-normality, which can lead to pronounced transient growth even when the system is asymptotically stable [47].

Examples of radial matrices include normal, Hermitian (symmetric), and skew-Hermitian (skew-symmetric) matrices. A matrix Q is *normal* if $QQ^* = Q^*Q$, which guarantees that its eigenvectors form an orthonormal basis. A matrix is *Hermitian* if $Q^* = Q$ (or symmetric if $Q^\top = Q$), and such matrices have real eigenvalues and orthonormal eigenvectors. Conversely, a matrix is *skew-Hermitian* if $Q^* = -Q$, in which case its eigenvalues are purely imaginary. For Hermitian (symmetric) matrices, the numerical range lies on the real axis, while for skew-Hermitian (skew-symmetric) matrices, it lies on the imaginary axis.

More generally, for any matrix Q, its Hermitian $H(Q)$ and skew-Hermitian $S(Q)$ parts satisfy $\omega(H(Q)) = \Re(\omega(Q))$, and $\omega(S(Q)) = \Im(\omega(Q))$. The numerical radius can be characterized using the Hermitian part of rotated matrices. Specifically,

$$r(Q) = \max_{\vartheta\in\mathbb{R}} \lambda_{\max}\big(H(e^{i\vartheta}Q)\big).$$

The numerical range of the product $e^{i\vartheta}Q$, i.e., $\omega(e^{i\vartheta}Q)$, is the same as $\omega(Q)$ rotated around the origin by an angle ϑ. This characterization also provides equivalent conditions for bounding the numerical radius, which are particularly useful for analyzing stability in DT systems. These conditions are summarized in the following lemma.

Lemma 2.2 ([44]). *Let $Q \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

(i) $r(Q) \le 1$;

(ii) $\rho\big(H(e^{i\theta}Q)\big) \le 1$;

(iii) $\lambda_{\max}\big(H(e^{i\theta}Q)\big) \le 1$;

(iv) $\|H(e^{i\theta}Q)\|_2 \le 1$

for all $\theta \in \mathbb{R}$.

Diagonal Dominance. A matrix $Q = [Q_{ij}] \in \mathbb{C}^{n\times n}$ is *diagonally dominant* if the magnitude of each diagonal entry is greater than or equal to the sum of the magnitudes of the other entries in its row:

$$|Q_{ii}| \ge \sum_{j\neq i} |Q_{ij}|, \quad \text{for all } i = 1, \dots, n.$$

This property is central to numerical analysis, particularly in guaranteeing the convergence of iterative methods for solving linear systems, and it often implies stability. Different types of diagonal dominance are defined as follows. Q is *strictly diagonally dominant* if the inequality is strict for all rows: $|Q_{ii}| > \sum_{j\neq i} |Q_{ij}|$, for all $i = 1, \ldots, n$. Q is *irreducibly diagonally dominant* if Q is irreducible, diagonally dominant, and the strict inequality $|Q_{ii}| > \sum_{j\neq i} |Q_{ij}|$ holds for at least one row i [43, Corollary 6.2.27]. Lastly, Q is *diagonally equipotent* if equality holds for all rows: $|Q_{ii}| = \sum_{j\neq i} |Q_{ij}|$, for all $i = 1, \ldots, n$.

These classes have strong implications for the matrix's eigenvalues and nonsingularity. Matrices that are strictly or irreducibly diagonally dominant are nonsingular. Furthermore, if all diagonal elements Q_{ii} are strictly positive, then all eigenvalues of strictly or irreducibly diagonally dominant matrices have positive real parts [43, Theorem 6.1.10].

For the equipotent case, if Q is irreducible, diagonally equipotent, and has nonnegative diagonal elements ($Q_{ii} \geq 0$), then its eigenvalues have nonnegative real parts, and its corank is at most 1.

2.2 Perron–Frobenius Theory

PF theory establishes fundamental spectral properties for a special class of matrices. The classical theory applies to positive or irreducible nonnegative matrices—those with strictly positive entries or nonnegative entries that preserve strong connectivity. For such matrices, the eigenvalue corresponding to the spectral radius is real, positive, and simple, and the associated left and right eigenvectors are strictly positive. This theory was later extended to eventually positive (EP) matrices, where some power of the matrix becomes strictly positive, even if the original matrix contains negative entries. Beyond this, we define the strong PF property for an even broader class of matrices. This condition requires only that the spectral radius is a simple, real, and positive eigenvalue that strictly dominates the rest of the spectrum, and that the associated right eigenvector is positive. These properties are discussed in detail in this section.

Positive and Nonnegative Matrices. A matrix $Q = [Q_{ij}] \in \mathbb{R}^{n\times n}$ is called *positive*, denoted $Q > 0$, if $Q_{ij} > 0$ for all i, j. It is *nonnegative*, denoted $Q \geq 0$, if $Q_{ij} \geq 0$ for all i, j. For positive matrices, the spectral radius governs the asymptotic action of the matrix via a unique positive direction. The next result characterizes this property and forms the basis for many extensions of PF theory.

Theorem 2.1 (Perron's Theorem [43, Theorem 8.2.8]). *Let $Q \in \mathbb{R}^{n\times n}$ be a positive matrix. Then:*

(i) $\exists\ \lambda_1 \in \Lambda(Q)$ *such that* $\lambda_1 = \rho(Q)$, $\lambda_1 > 0$, λ_1 *is real and simple;*

(ii) $\exists$ *a unique (up to scaling) vector* $\mathbf{v} > 0$ *satisfying* $Q\mathbf{v} = \lambda_1\mathbf{v}$*;*

(iii) $\exists$ *a unique (up to scaling) vector* $\mathbf{w} > 0$ *satisfying* $\mathbf{w}^\top Q = \lambda_1\mathbf{w}^\top$*;*

(iv) λ_1 *is strictly dominant, i.e.,* $|\lambda_i| < \lambda_1$ *for* $i \geq 2$*;*

(v) The normalized powers of Q *converge to a rank-one matrix:*

$$\lim_{k\to\infty} \frac{Q^k}{\lambda_1^k} = \mathbf{v}\mathbf{w}^\top.$$

For nonnegative matrices, the first three properties (i)-(iii) of Theorem 2.1 hold in a weaker form: the dominant eigenvalue satisfies $\lambda_1 \geq 0$, and the corresponding right and left eigenvectors satisfy $\mathbf{v} \geq 0$ and $\mathbf{w} \geq 0$. When the matrix is additionally *irreducible*, these inequalities become strict, yielding $\lambda_1 > 0$ and $\mathbf{v}, \mathbf{w} > 0$. This refinement is captured by the PF theorem stated below.

Theorem 2.2 (PF Theorem [43, Theorem 8.4.4]). *Let* $Q \in \mathbb{R}^{n\times n}$ *be nonnegative and irreducible. Then:*

(i) $\exists$ *a real, positive, and simple* $\lambda_1 \in \Lambda(Q)$ *such that* $\lambda_1 = \rho(Q)$*;*

(ii) $\exists$ *a unique (up to scaling) vector* $\mathbf{v} > 0$ *satisfying* $Q\mathbf{v} = \lambda_1\mathbf{v}$*;*

(iii) $\exists$ *a unique (up to scaling) vector* $\mathbf{w} > 0$ *satisfying* $\mathbf{w}^\top Q = \lambda_1\mathbf{w}^\top$*.*

The key difference between positive and irreducible nonnegative matrices is that, in the latter case, λ_1 need not be strictly dominant. For example, consider

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is nonnegative and irreducible, yet its eigenvalues are $\{1, -1\}$, so $|\lambda_2| = \lambda_1$.

To ensure strict spectral separation, the notion of *primitivity* is introduced. A nonnegative matrix Q is called primitive if it is irreducible and its spectral radius is strictly greater than the modulus of all other eigenvalues, i.e., $\lambda_1 = \rho(Q) > |\lambda_i|$ for all $\lambda_i \in \Lambda(Q)$, $i \geq 2$. The class of primitive matrices is particularly relevant as it guarantees that properties (iv)–(v) of Perron's Theorem 2.1 hold when $Q \geq 0$.

Strong Perron–Frobenius Theory and Eventually Positive Matrices. The classical PF theory guarantees strong spectral properties for positive and irreducible nonnegative matrices. However, many real-world systems involve antagonistic or mixed interactions, leading to signed matrices. To extend PF theory to such settings, we first define the *strong (right) PF property*, denoted $Q \in \mathcal{PF}$. This property requires that the eigenvalue associated with the spectral radius $\rho(Q)$ is real, positive, algebraically simple, strictly dominant over all others, and that the corresponding right eigenvector $\mathbf{v}$ is positive.

Among matrices that satisfy the strong PF property, one important class is that of *eventually positive* (EP) matrices. A matrix $Q \in \mathbb{R}^{n\times n}$ is called EP, denoted $Q \overset{\vee}{>} 0$, if $\exists\ k_0 \in \mathbb{N}$ such that $Q^k > 0$ for all $k \geq k_0$. EP requires that both Q and its transpose $Q^\top$ satisfy the strong PF property, as shown in the following lemma.

Lemma 2.3 ([48, Theorem 2.2]). *Let $Q \in \mathbb{R}^{n\times n}$. The following statements are equivalent:*

(i) $Q \in \mathcal{PF}$ *and* $Q^\top \in \mathcal{PF}$;

(ii) $Q \overset{\vee}{>} 0$;

(iii) $Q^\top \overset{\vee}{>} 0$.

Eventually Exponentially Positive Matrices. Analogous to the concept of EP in DT, we define *Eventually Exponentially Positive* (EEP) matrices for the CT setting. A matrix $Q \in \mathbb{R}^{n\times n}$ is called EEP if there exists a time $t_0 \geq 0$ such that the matrix exponential satisfies $e^{Qt} > 0$ for all $t \geq t_0$.

There is a direct structural link between the EP and EEP classes. Specifically, a matrix Q is EEP *iff* $\exists$ a scalar $d \in \mathbb{R}$ such that the shifted matrix $Q + dI \overset{\vee}{>} 0$. Consequently, for Q EEP, its spectral abscissa $\alpha(Q) = \max_i \Re(\lambda_i(Q))$ is a simple, nonnegative, and real eigenvalue, and the associated left and right eigenvectors are strictly positive.

M-matrices. An important class of matrices closely related to PF theory is that of M-matrices. A matrix $Q \in \mathbb{R}^{n\times n}$ is classified as a *Z-matrix* if all of its off-diagonal entries are non-positive, i.e., $Q_{ij} \leq 0$ for all $i \neq j$. This structure is fundamental to the definition of M-matrices.

A matrix Q is an *M-matrix* if it is a Z-matrix and all its eigenvalues have nonnegative real parts. A common and useful alternative definition is that a Z-matrix Q is an M-matrix if it can be expressed in the form $Q = sI - R$, where $R \geq 0$ and the scalar s satisfies $s \geq \rho(R)$. When $s > \rho(R)$, the M-matrix is nonsingular. The following lemma provides several equivalent conditions for a Z-matrix to be a nonsingular M-matrix.

Lemma 2.4 ([49]). *For a Z-matrix $Q = sI - R$ with $s \geq 0$ and $R \geq 0$, the following conditions are equivalent to Q being a nonsingular M-matrix:*

(i) All eigenvalues of Q have positive real parts.

(ii) Q is nonsingular and $Q^{-1} \geq 0$.

(iii) $\exists$ a vector $\mathbf{x} > 0$ such that $Q\mathbf{x} > 0$.

When $s = \rho(R)$, the matrix is singular. The following lemma provides the properties of a singular M-matrix, specifically for the irreducible case.

Lemma 2.5 ([50]). *Let $Q = sI - R$ with $s \geq 0$ and $R \geq 0$ be an irreducible Z-matrix. The following conditions are equivalent to Q being a singular M-matrix:*

(i) $0 \in \Lambda(Q)$ *is a simple eigenvalue, and all other eigenvalues have positive real parts.*

(ii) $\exists$ *a unique (up to scaling) vector* $\mathbf{v} > 0$ *such that* $Q\mathbf{v} = \mathbf{0}$.

(iii) $\exists$ *a unique (up to scaling) vector* $\mathbf{w} > 0$ *such that* $\mathbf{w}^\top Q = \mathbf{0}^\top$.

Remark 2.1. The properties of singular M-matrices are particularly relevant in graph theory, where the graph Laplacian L of a strongly connected, undirected graph is an irreducible, singular M-matrix. In this case, the corank of L is 1, and the null space is spanned by the all-ones vector $\mathbb{1}$, which corresponds to the consensus state.

3

Graph Theory

Graph theory offers a rigorous framework for modeling interactions in multi-agent systems, such as social networks. In these settings, graphs encode not only who interacts with whom, but also the strength and nature—cooperative or antagonistic—of these relationships. This chapter introduces the graph-theoretic tools essential for the analysis developed throughout the thesis, with particular emphasis on adjacency and Laplacian matrices for directed and signed graphs.

3.1 Graphs

A graph is defined as a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges, and $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the adjacency matrix associated with $\mathcal{G}$. We allow weighted edges, i.e., $a_{ij} > 0$ *iff* there exists a directed edge from node j to node i, i.e., $(j, i) \in \mathcal{E}$. A graph is undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$, and $A^\top = A$. In a directed graph (digraph), $(i, j) \in \mathcal{E}$ does not imply $(j, i) \in \mathcal{E}$, and/or $a_{ij} \neq a_{ji}$. A graph is said to be unweighted if $a_{ij} \in \{0, 1\}$, and has no self loops if $a_{ii} = 0$ for all i. In this thesis, we study weighted digraphs without self loops.

For a digraph, the *neighbors* of a node are classified according to edge direction. The *in-neighbors* of node i are the set of nodes that have edges pointing to i, i.e., $\mathcal{N}_{\text{in}}(i) = \{\, j \in \mathcal{V} \mid (j, i) \in \mathcal{E} \,\}$. Similarly, the *out-neighbors* of node i are the set of nodes that i points to: $\mathcal{N}_{\text{out}}(i) = \{\, j \in \mathcal{V} \mid (i, j) \in \mathcal{E} \,\}$. The cardinalities of these sets correspond to the *in-degree* and *out-degree* in the unweighted case. For weighted graphs, the in-degree of a node i is defined as the sum of the weights of the edges incoming to i, given by $\xi_{\text{in},i} = \sum_j a_{ij}$, and the diagonal matrix of in-degrees is denoted by $\Sigma_{\text{in}} = \text{diag}(\boldsymbol{\xi}_{\text{in}})$. Similarly, the out-degree of node j is defined as

$\xi_{\text{out},j} = \sum_i a_{ij}$, with $\Sigma_{\text{out}} = \text{diag}(\boldsymbol{\xi}_{\text{out}})$. A digraph is said to be *weight-balanced* if the total inflow equals the total outflow at each node, i.e., $A\mathbb{1} = A^\top \mathbb{1}$.

A *walk* of length k from node i to node ℓ is a sequence of adjacent nodes $i = v_0 \to v_1 \to \cdots \to v_k = \ell$. A *path* is a walk in which all nodes are distinct, and a *cycle* is a closed path, meaning the starting and ending node coincide. Node i is said to *reach* node ℓ if there exists a path from i to ℓ.

A digraph is *strongly connected* if every node can reach every other node; equivalently, the adjacency matrix A is irreducible. A digraph is *unilaterally connected* if, for every pair of nodes (i, j), there exists a directed path from i to j or from j to i (but not necessarily both). A digraph is *weakly connected* if its underlying undirected graph—obtained by ignoring edge directions—is connected, and it is *not connected* if this condition fails. A subgraph that is strongly connected is called a *strongly connected component* (SCC).

A *directed spanning tree* is a directed, cycle-free subgraph in which one node (called the root) has a directed path to all other nodes. The existence of such a tree is a weaker condition than strong connectivity.

3.1.1 Laplacian Matrices

The Laplacian matrix is a fundamental operator that provides an algebraic representation of a graph's connectivity. Its primary role is to couple the graph's structure to the dynamics of processes unfolding on it, such as consensus, diffusion, or synchronization. For a directed graph $\mathcal{G}$, the primary definition used in this thesis is the *in-degree Laplacian*, defined as $L = \Sigma_{\text{in}} - A$, where Σ_{in} is the diagonal matrix of in-degrees. Element-wise, the entries are given by:

$$[L]_{ij} = \begin{cases} \sum_{k \neq i} a_{ik}, & \text{if } j = i, \\ -a_{ij}, & \text{if } j \neq i. \end{cases} \tag{3.1}$$

By construction, this matrix satisfies the zero row-sum property, $L\mathbb{1} = \mathbf{0}$. This definition is chosen because it naturally models the flow of information in consensus dynamics. In the CT consensus model $\dot{\mathbf{x}} = -L\mathbf{x}$, the state evolution of agent i is driven by the weighted sum of differences relative to its neighbors: $\dot{x}_i = -\sum_{j \neq i} a_{ij}(x_i - x_j)$. An alternative, the *out-degree Laplacian* ($L' = \Sigma_{\text{out}} - A^\top$), models information outflow. The two constructions are dual: the in-degree Laplacian of a graph $\mathcal{G}$ is the out-degree Laplacian of the graph $\mathcal{G}'$ obtained by reversing all edge directions.

A related and widely used variation is the *random-walk (normalized) Laplacian*, which normalizes the weights by the node degrees. It is defined as

$$L_{\text{rw}} = \Sigma_{\text{in}}^{-1} L = I - \Sigma_{\text{in}}^{-1} A = I - W,$$

where $W = \Sigma_{\text{in}}^{-1} A$ is the row-stochastic transition matrix of a random walk on the graph. While L captures the absolute flow of influence, L_{rw} captures the relative influence and is central to the study of Markov chains.

The fundamental properties of Laplacian L are well-known and summarized in the following lemmas [28, 51–56].

Lemma 3.1 ([28]). *The Laplacian L of a digraph $\mathcal{G}$ has the following properties:*

(i) $L\mathbb{1} = \mathbf{0}$. *Thus,* $0 \in \Lambda(L)$ *with* $\mathbb{1}$ *as a corresponding right eigenvector.*

(ii) All eigenvalues of L have non-negative real parts, i.e., $\Re[\lambda] \geq 0 \; \forall \, \lambda \in \Lambda(L)$.

(iii) L is a singular M-matrix.

The properties of the zero eigenvalue are directly related to the graph's connectivity, specifically its strong connectivity.

Lemma 3.2 ([28]). *If the digraph $\mathcal{G}$ is strongly connected, its Laplacian L has the following properties:*

(i) L is a singular and irreducible M-matrix.

(ii) $0 \in \Lambda(L)$ *is simple. Consequently,* $\operatorname{rank}(L) = n - 1$ *and* $\ker(L) = \operatorname{span}\{\mathbb{1}\}$.

(iii) $\exists$ *unique (up to scaling) eigenvector* $\mathbf{w} > 0$ *such that* $\mathbf{w}^\top L = \mathbf{0}^\top$ *and* $\ker(L^\top) = \operatorname{span}\{\mathbf{w}\}$.

(iv) This vector $\mathbf{w}$ *is also the unique vector (up to scaling) such that the matrix* $\Pi L + L^\top \Pi \succeq 0$, *with* $\Pi := \operatorname{diag}\{\mathbf{w}\}$, *and* $\operatorname{rank}(\Pi L + L^\top \Pi) = n - 1$.

For weight balanced graphs, we obtain a stronger result as follows.

Lemma 3.3 ([28]). *Let $\mathcal{G}$ be a strongly connected digraph. The following statements are equivalent:*

(i) $\mathcal{G}$ is weight-balanced, i.e., $A\mathbb{1} = A^\top \mathbb{1}$ *(or* $\Sigma_{\text{in}} = \Sigma_{\text{out}}$*).*

(ii) $L^\top \mathbb{1} = \mathbf{0}$. *Consequently,* $\ker(L) = \ker(L^\top) = \operatorname{span}\{\mathbb{1}\}$.

(iii) The symmetric part of L, $L_s = \frac{L + L^\top}{2} \succeq 0$, *and* $\operatorname{rank}(L_s) = n - 1$.

3.2 Signed Graphs

A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ extends the standard graph model by allowing edge weights to be negative, i.e., $a_{ij} \gtreqless 0$. This provides a framework for modeling both cooperative (positive, i.e. $a_{ij} > 0$) and antagonistic (negative, i.e., $a_{ij} < 0$) relationships within a network. In the signed case, unweighted adjacency matrix has entries $a_{ij} \in \{-1, 0, 1\}$ and weighted otherwise.

A fundamental notion for signed graphs is *structural balance*. A signed graph is structurally balanced if its nodes can be partitioned into two sets, $\mathcal{V}_1$ and $\mathcal{V}_2$, such that all edges within a set are positive ($a_{ij} \geq 0$ for $i, j \in \mathcal{V}_i$) and all edges

between sets are negative ($a_{ij} \leq 0$ for $i \in \mathcal{V}_i$ and $j \in \mathcal{V}_j$). Equivalently, a graph is structurally balanced if $\exists$ a diagonal *signature* (or *gauge*) matrix $\Xi = \text{diag}(v_i)$, with $v_i \in \{+1, -1\}$, such that the "gauged" adjacency matrix $\Xi A \Xi$ is non-negative (i.e., $(\Xi A \Xi)_{ij} \geq 0$). This transformation corresponds to a change of variables $\mathbf{z} = \Xi \mathbf{x}$ that "flips the sign" of edges between the partitions. This equivalence characterizes balanced networks as those that are "gauge-equivalent" to purely cooperative ones. When balance fails (unbalanced graphs), this antagonism cannot be removed globally.

3.2.1 Opposing vs. Repelling Laplacians

In the literature, the term "signed Laplacian" can refer to two distinct matrix constructions that model antagonism in different ways [24, 25].

The first is the *opposing signed Laplacian*, L_o, proposed in [23] and defined element-wise as

$$[L_o]_{ij} = \begin{cases} \sum_{k \neq i} |a_{ik}|, & \text{if } j = i, \\ -a_{ij}, & \text{if } j \neq i. \end{cases}$$

In compact form, $L_o = \Sigma_{|\text{in}|} - A$, where $\Sigma_{|\text{in}|} = \text{diag}(\boldsymbol{\xi}_{|\text{in}|})$ and $\xi_{|\text{in}|,i} = \sum_{j \neq i} |a_{ij}|$.

This Laplacian has several strong properties. It is diagonally equipotent, which ensures that all its eigenvalues have non-negative real parts. A key distinction from the unsigned Laplacian is that 0 is a simple eigenvalue of L_o if and only if the graph is structurally balanced and the gauge-transformed graph (with adjacency matrix $\Xi A \Xi$) is strongly connected. In this balanced case, the kernel is spanned by a bipartite vector $\mathbf{v}$ (with entries $v_i \in \{\pm 1\}$), i.e., $L_o \mathbf{v} = \mathbf{0}$, and a corresponding left eigenvector $\mathbf{w} \gneqq \mathbf{0}$ exists such that $\mathbf{w}^\top L_o = \mathbf{0}^\top$. The eigenvectors are typically normalized such that $\mathbf{w}^\top \mathbf{v} = 1$, and the state transition matrix converges to a rank-one matrix: $\lim_{t \to \infty} e^{-L_o t} = \mathbf{v}\mathbf{w}^\top$. If the graph is not structurally balanced, then 0 is not an eigenvalue, all eigenvalues of L_o have positive real parts, and the state transition matrix converges to a null matrix: $\lim_{t \to \infty} e^{-L_o t} = \mathbf{O}$.

The second, and the one used throughout this thesis, is the *repelling signed Laplacian*, L_r, defined in [25] as

$$[L_r]_{ij} = \begin{cases} \sum_{k \neq i} a_{ik}, & \text{if } j = i, \\ -a_{ij}, & \text{if } j \neq i. \end{cases}$$

This is precisely the same in-degree construction (3.1) used for unsigned graphs, where the in-degree is the algebraic sum of incoming weights: $L_r = \Sigma_{\text{in}} - A$.

Henceforth, we will refer to the repelling Laplacian L_r as L. The key property $L\mathbb{1} = \mathbf{0}$ is preserved, so 0 is always an eigenvalue. However, unlike the unsigned case, the zero eigenvalue is not necessarily simple, and its multiplicity can be greater than one. The left kernel $\ker(L^\top)$ is spanned by one or more left eigenvectors $\mathbf{w}$ (satisfying $\mathbf{w}^\top L = \mathbf{0}^\top$), which may contain positive, negative, or zero entries. In contrast to the unsigned case, L is not an M-matrix or diagonally equipotent.

Crucially, because L is not diagonally equipotent, its nonzero eigenvalues are not guaranteed to have positive real parts. Consequently, the repelling Laplacian can lead to unstable dynamics where opinions diverge rather than converge. Furthermore, as noted, the zero eigenvalue may not be simple, which can prevent the formation of a global consensus.

To identify the subset of signed interaction topologies that admit stable consensus dynamics, we restrict our analysis to signed graphs whose negative Laplacian $-L$ is EEP, as formalized in Section 2.2. Specifically, if $-L$ is EEP, the following conditions are satisfied:

(i) $L\mathbb{1} = \mathbf{0}$ and $0 \in \Lambda(L)$ is a simple eigenvalue.

(ii) $\exists$ a strictly positive left eigenvector $\mathbf{w} > \mathbf{0}$ associated with the zero eigenvalue, i.e., $\mathbf{w}^\top L = \mathbf{0}^\top$.

(iii) All nonzero eigenvalues have positive real parts, i.e., $\Re[\lambda] > 0$ for all $\lambda \in \Lambda(L) \setminus \{0\}$.

(iv) The Lyapunov condition $\Pi L + L^\top \Pi \succeq 0$ is satisfied for $\Pi = \mathrm{diag}(\mathbf{w}) \succ 0$.

The primary dynamical implication of $-L$ being EEP is the convergence of the state transition matrix to a rank-one positive matrix: $\lim_{t\to\infty} e^{-Lt} = \mathbb{1}\mathbf{w}^\top$, assuming the normalization $\mathbf{w}^\top\mathbb{1} = 1$.

3.2.2 Interaction Matrices

While Laplacians are central to some models, *interaction matrices*, denoted W, are fundamental to others, particularly in areas like Markov chains to represent the flow of information or probability.

Interaction matrices can be generated in several ways. One common method is the *random-walk transition matrix*, $W_{\mathrm{rw}} = \Sigma_{\mathrm{in}}^{-1} A$, which assumes an unsigned graph ($A \geq 0$). This matrix is row-stochastic ($W_{\mathrm{rw}}\mathbb{1} = \mathbb{1}$) and has zero diagonal entries, implying no self-loops.

In this thesis, we deal with a broader class of interaction matrices, the *averaging interaction matrix*, which is constructed from a Laplacian L as $W = I - \varepsilon L$, where $\varepsilon > 0$ is a small positive scalar, typically chosen as $\varepsilon < 1/\max_i(\xi_{\mathrm{in},i})$ to ensure W is non-negative (in the unsigned case). This construction is more general as it naturally includes self-loops (i.e., $W_{ii} > 0$).

For an unsigned graph, the resulting matrix $W = I - \varepsilon L$ is row-stochastic. The eigenvalue of W at 1 has a multiplicity equal to the number of strongly connected components of the graph and all other eigenvalues have magnitude less than 1. If the graph is strongly connected, then 1 is a simple and dominant eigenvalue of W, and its corresponding left eigenvector $\mathbf{w}$ is positive. In this case, the powers of W converge to a rank-one matrix: $\lim_{k\to\infty} W^k = \mathbb{1}\mathbf{w}^\top$, assuming $\mathbf{w}^\top\mathbb{1} = 1$.

For signed graphs, this construction leads to two different interaction matrices, corresponding to the opposing and repelling Laplacians:

- **Opposing Interaction Matrix.** Using the opposing Laplacian L_o, we get $W_o = I - \varepsilon L_o$. In this case, the absolute value of the interaction matrix is row-stochastic ($|W_o|\mathbb{1} = \mathbb{1}$). If the graph is structurally balanced and the gauge-transformed graph ($\Xi W \Xi$) is strongly connected, then for a sufficiently small ε, 1 is a simple eigenvalue of W_o and all other eigenvalues $\lambda \in \Lambda(W_o) \setminus \{1\}$ have magnitude less than 1 ($|\lambda| < 1$). In this case, $W_o\mathbf{v} = \mathbf{v}$ and powers of W_o converge to the rank-one matrix: $\lim_{k\to\infty} W_o^k = \mathbf{v}\mathbf{w}^\top$ where $\mathbf{w}^\top W_o = \mathbf{w}^\top$ with $\mathbf{w} \gneqq 0$, and normalized such that $\mathbf{w}^\top\mathbf{v} = 1$. If it is not structurally balanced, then $\rho(W_o) < 1$ and powers of W_o converge to the null matrix: $\lim_{k\to\infty} W_o^k = \mathbf{O}$.
- **Repelling Interaction Matrix.** Using the repelling Laplacian L_r, we get $W_r = I - \varepsilon L_r$. This is the primary form used in this thesis. This matrix is row-sum one ($W_r\mathbb{1} = \mathbb{1}$), so 1 is always an eigenvalue. However, it does not share the stability properties of the other cases: the spectral radius of W_r, $\rho(W_r)$, is not necessarily 1 and can be greater than 1. Furthermore, the eigenvalue at 1 is not necessarily simple; its multiplicity can be greater than one. The corresponding left eigenvector(s) $\mathbf{w}$ (satisfying $\mathbf{w}^\top W_r = \mathbf{w}^\top$) may contain positive, negative, or zero entries. Consequently, the powers of W_r may diverge.

Eventually Stochastic Matrices. A special case of repelling interaction matrix, referred to as *eventual stochastic* (ES) matrix, is of particular interest. In this case, despite possibly having negative entries, W_r has 1 as a simple and strictly dominant eigenvalue (i.e., $1 \in \Lambda(W_r)$ and all other eigenvalues have magnitude less than 1), with a corresponding positive dominant left eigenvector $\mathbf{w} > \mathbf{0}$. In this case, $\lim_{k\to\infty} W_r^k = \mathbb{1}\mathbf{w}^\top$ such that $\mathbf{w}^\top\mathbb{1} = 1$.

4

Concluding Remarks

The focus of this thesis is the study of opinion dynamics in social networks, with a specific emphasis on understanding how collective behavior is influenced by the presence of antagonistic interactions. While classical models typically assume cooperative environments where opinions remain within the convex hull of initial conditions, this work systematically explores scenarios where rivalry, distrust, or opposition—encoded via signed graphs—can lead to more complex phenomena such as transient divergence, and convex hull violation. A fundamental component of the dynamics studied here is that individuals exhibit stubbornness, manifesting as a persistent attachment to their initial opinions.

In this regard, the first paper of this thesis (Paper A) explores the SFJ model in which individuals exhibit stubbornness while interacting in an antagonistic network. This work moves beyond the standard opposing Laplacian formulation, which relies heavily on structural balance theory, to the repelling interaction mechanism in both DT and CT. It is shown that ES or EEP matrices, which imply stability for the DeGroot model, are insufficient for the stability of the SFJ model. Consequently, we establish that stability requires an additional sufficient condition: the existence of a dominant sharp point in the numerical range of the interaction matrix. A key finding is that, unlike in cooperative models, the equilibrium opinions in an SFJ model need not be confined to the convex hull of the initial conditions; agents may adopt positions more extreme than any initial view held in the network.

Furthermore, when these discussion events (modeled via the SFJ dynamics) are concatenated to represent repeated negotiations, the system may exhibit transient amplification. In this phenomenon, opinions temporarily move apart and distance themselves before eventually reconverging towards a consensus value within the convex hull of the initial opinions. This behavior is attributed to the non-normality

of the system matrices, which allows for transient growth even in stable systems. We also show that under certain conditions, the concatenated model can lead to opinion divergence, highlighting that stability in the SFJ model does not imply stability over repeated interactions of the SFJ model as in the concatenated SFJ model. We prove that the same spectral assumptions utilized for the single-round case (SFJ model) also provide sufficient conditions for the stability of the concatenated SFJ model.

The model is then extended to bipartite graphs and time-varying interaction structures. It is shown that, with gauge transformations, bipartite networks lead to a bipartite consensus (consensus in absolute values). For the time-varying case, we demonstrate that even if the conditions on the interaction matrices are satisfied at each step, arbitrary changes in the stubbornness values can lead to divergence. To have sufficient stability in this setting, we establish that stubbornness values must either remain constant over the discussion intervals or be uniform across all agents at every discussion. We also provide a common quadratic Lyapunov function formulation that ensures stability under arbitrary changes in both the interaction and stubbornness matrices.

The second class of models considered in this thesis work, introduced in Paper B, extends the SFJ framework to the multidimensional case, acknowledging that individuals often discuss multiple interdependent topics simultaneously. Here, the dynamics are governed by the interplay between the social network structure and a logical dependency matrix connecting the topics. It is shown that the stability of the collective system in the concatenated multidimensional SFJ model is not merely a superposition of independent topics; rather, the coupling between topics can destabilize an otherwise stable social network. Formally, sufficient conditions for convergence are derived and the analysis reveals that the concatenated multidimensional SFJ model may lead to diverse outcomes, ranging from full, partial, or bipartite consensus to divergence.

While Papers A and B examine the convergence properties of the underlying dynamics, Paper C shifts the focus to the quality of the collective decision, particularly in relation to the wisdom of crowds. This work extends the notion of collective accuracy to signed networks by establishing conditions under which social interactions either increase or decrease the variance of the group estimate around the true value. The analysis reveals that introducing antagonistic ties reshapes the geometry of the concentration region—the set of social power allocations that enhance wisdom. In unsigned networks, this region is confined to a subset of a simplex; however, in signed networks, it becomes a subset of a hyperplane that accommodates negative weights. Remarkably, even with negative social powers, the estimate remains centered around the true value in networks where consensus is achieved.

When the framework is extended to bipartite networks, a striking phenomenon emerges: an agreement does not imply accuracy. In such cases, a group may reduce its variance (become more confident) while converging toward an incorrect value, effectively becoming confidently wrong due to the structural properties of

the network. The paper further generalizes the analysis to scenarios where initial opinions are correlated rather than independent. Under correlation, the geometry of the concentration region changes significantly. Here, the optimal social power vector—the allocation that minimizes group variance—may lie outside the standard simplex, making signed networks (with negative weights) essential for achieving optimal wisdom. Additionally, we identify situations where the variance of the collective opinion cannot be reduced below the minimum variance of any individual agent, as well as cases where restricting social power to the simplex prevents the group from surpassing the wisdom of its most accurate member.

Directions of Future Research

This section discusses possible directions in which the results presented in this thesis could be extended.

Regarding the signed dynamics presented in Papers A and B, the current analysis primarily relies on interaction matrices that form a specific subset of weight-balanced graphs. A future research direction is to identify the necessary conditions for stability across a more general class of signed matrices. Specifically, it would be interesting to characterize broader families of interaction topologies for which any magnitude of stubbornness implies stability, thereby relaxing the structural constraints currently required. Additionally, it would be valuable to analyze the robustness of this stability with respect to perturbations in both the interaction weights and stubbornness parameters, determining the precise margins within which the system's convergence properties are preserved. Another direction is the rigorous quantification of transient amplification, where opinions temporarily exit beyond their initial bounds before eventually converging. While this thesis identifies that such behavior is driven by the non-normality of the system matrix, the necessary and sufficient conditions for this phenomenon remain an open problem.

Concerning the multidimensional framework in Paper B, the current model assumes a homogeneous logical structure in which all agents share the same dependency matrix among topics. A compelling extension would be to consider the heterogeneous case, where the topic dependency matrix varies across individuals, reflecting diverse cognitive models. Additionally, making the topic dependency matrix adaptive would allow for modeling real human psychology. For example, as a group gets closer to an agreement, people often feel less internal conflict and become more willing to compromise on other related issues to fit in.

Finally, the analysis of the wisdom of crowds in Paper C motivates a transition from descriptive analysis—simply observing how opinions evolve—to prescriptive modeling, where the goal is to actively design networks that maximize accuracy. A key direction is the development of decentralized mechanisms for selecting parameters, such as interaction weights or stubbornness, to improve collective wisdom without central coordination. In this context, a game-theoretic framework can be developed in which individuals strategically adjust their parameters to maximize their own influence or accuracy. Furthermore, the finding that variance may transiently decrease before increasing highlights the necessity of developing finite-time

performance metrics and optimal stopping criteria to capture collective accuracy before polarization degrades the outcome.

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