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Theoretical Considerations and Foundations for the Erasmus+ Cooperation Partnership *dim²ension*: A Project Report

Jonas Bergman Ärlebäck, Simon Barlovits, Georgios Fesakis, Alissa Fock, Sara Julsgård, Dimitrios Lappas, Matthias Ludwig, Maria-Anastasia Moustaka, Konstantinos Perdikaris, Hans-Stefan Siller

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Theoretical Considerations and Foundations for the Erasmus+ Cooperation Partnership *dim²ension*: A Project Report

***Di**gital Supported **M**athematical **M**od**e**lli**n**g*
*For **S**ustainable Development Goals **I**n Eur**o**pean Educatio**n***

Authors

Jonas Bergman Ärlebäck[1], Simon Barlovits[2], Georgios Fesakis[3], Alissa Fock[4], Sara Julsgård[1], Dimitrios Lappas[3], Matthias Ludwig[2], Maria-Anastasia Moustaka[3], Konstantinos Perdikaris[3], & Hans-Stefan Siller[4]

Reviewers

Susana Carreira[5,6], Hélia Jacinto[5], Claudia Lázaro del Pozo[7], Stavroula Prantsoudi[3], & Juan Antonio Reyes Delgado[7]

[1]Linköping University, Sweden;
[2]Goethe University Frankfurt, Germany;
[3]University of the Aegean, Greece;
[4]Julius Maximilian University of Würzburg, Germany;
[5]University of Lisbon, Portugal;
[6]University of Algarve, Portugal;
[7]Spanish Federation of Mathematics Teacher's Societies, Spain

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1. Introduction, aim and purpose

The *dim²ension* project aims to integrate digital tools into mathematical modelling to address real-world problems, particularly those related to the United Nations' Sustainable Development Goals (SDGs). The project's overarching objective is twofold: firstly, to enhance students' mathematical modelling competencies, and secondly, to foster awareness of global sustainability issues in schools. The present introductory chapter offers a comprehensive overview of the project, while also delineating the structure of the theoretical white paper.

This theoretical white paper provides a concise overview of the theoretical scaffold (cf. Lester, 2005) that serves as the foundation for the *dim²ension* project. The document delineates the frame of reference that has been established by the project partners, encompassing the provision of precise definitions and a framework for task classification and task design criteria. In the context of scientific work, discussions and debates, it is imperative to have clear and precise definitions of the concepts involved. Indeed, as Niss (1996) noted, "[t]erminological issues are mostly tedious and boring. Additionally, this document serves as a foundation for ongoing discussions and refinements within the project, ensuring that all theoretical constructs align with the practical implementation of digital tools in mathematical modelling. Nevertheless, for a scholarly or scientific discourse to be serious, in fact possible, it is essential that at least the key entities and concepts of that discourse are reasonably clear for those involved. This is particularly true with a field like mathematics education in which transparency and clarity are not easily achieved, let alone to be taken as a matter of course." (p. 12).

1.1 The *dim²ension* project

The *dim²ension* project, which is funded by the European Union (grant number 2024-1-DE01-KA220-HED-000245297), focuses on the digital support of mathematical modelling activities, with a particular emphasis on integrating sustainability into mathematics education. The acronym *dim²ension* stands for '**Di**gital Supported **M**athematical **M**od**e**lli**n**g For **S**ustainable Development Goals **I**n Eur**o**pean Educatio**n**.' The project involves seven partners from five European countries and is scheduled to run from September 2024 to August 2027. The project's partners are as follows:

- Goethe University Frankfurt (Coordinator) [Germany]
- Autentek Development [Germany]
- Julius Maximilian University of Würzburg [Germany]
- Linköping University [Sweden]
- Spanish Federation of Mathematics Teacher's Societies [Spain]
- University of the Aegean [Greece]
- University of Lisbon [Portugal]

These institutions contribute a diverse range of expertise, including mathematics education research and software development, thereby ensuring a multidisciplinary approach to achieving the project's objectives.

The fundamental idea underpinning the project is the utilisation of mathematical modelling, predicated on the premise of investigating real-world phenomena though the application of mathematical knowledge and mathematical principles (Blum, 2015). Within the project, mathematical modelling is defined as the process of applying mathematical knowledge to describe, analyse or evaluate real-world situations (Blum & Niss, 1991). Consequently, mathematical modelling offers considerable potential for integrating mathematics with other STE(A)M disciplines and paving the way for addressing sustainability issues in mathematics education (Goos et al., 2023; Maas et al., 2019; Siller et al., 2024a, c). This report presents a framework within which these issues can be approached in general, and how this has been done in the *dim²ension* project in particular.

1.1.1 Brief background and project idea

We live in a time which is characterised by a series of global challenges, including climate change, food insecurity, armed conflict and economic instability. Consequently, it is imperative to recognise the necessity to confront these global challenges in this era of turbulence, which pervades all sectors, including politics, society, the economy, and education. In this regard, the education sector, as stated by UNESCO (2015), is required to facilitate learners in comprehending, evaluating and countering global challenges.

In the context of mathematics education, mathematical modelling in particular offers valuable opportunities to address these global challenges in the classroom setting (Maas et al., 2019; Siller et al., 2024b). Modelling, defined as the investigation of real-world phenomena using mathematics (Blum, 2015), is considered a central goal of mathematics education worldwide (Kaiser, 2020). In the context of the aforementioned global challenges, mathematical modelling facilitates the understanding and evaluation of phenomena from a mathematical perspective. Furthermore, it facilitates up the exploration of future pathways, i.e. the manner in which individuals or society should respond to global challenges (cf. UNESCO, 2015). Consequently, *from a didactic perspective*, mathematical modelling is considered to have great potential for integrating Education for Sustainable Development into mathematics education (cf. Siller et al., 2025a, 2025b), which in turn can be seen as a promising step towards promoting responsible citizenship (Maas et al., 2019; Siller et al., 2024a).

However, from a *praxis-oriented point of view*, the implementation of mathematical modelling activities in classroom practice remains challenging (Kaiser, 2020). Challenges perceived and encountered by teachers are for example difficulties when creating mathematical modelling tasks, diagnosing students' difficulties and formatively assessing students' mathematical modelling activities (Borromeo Ferri & Blum, 2010; Jablonski et al., submitted; Wess et al., 2021). Learners, on the other hand, encounter various challenges that are well-documented in the literature, such as working with open-ended tasks, identifying suitable models applicable to the real situations and validating the computed solutions (Blum, 2015; Galbraith & Stillman, 2006; Klock & Siller, 2020).

In order to address the present challenges in the teaching and learning of mathematical modelling, recent research has described the broad potential of digital tools to support mathematical modelling activities. These tools include the provision of dynamic representations or simulations, and the serving of as a resource for data collection and processing (cf. Cevikbas et al., 2023; Molina-Toro et al., 2019; Siller et al., 2023a). Hence, with the aim to exploit this potential of usage of digital tools to address the challenges associated with mathematical modelling, a digital tool is being developed to support mathematical modelling activities within the three-year *dim²ension* project (cf. Barlovits et al., submitted). More specifically, the *technical project outcome* is the development of a user-friendly digital tool for the preparation, implementation, management, monitoring and evaluation of mathematical modelling activities in educational practice.

In addition, the outcome of the *dim²ension* project will consist of *material-based*, *educational*, and *research-based outcomes.* Specifically, the *material-based project outcome* will be a rich, open-accessible database of high-quality mathematical modelling tasks. These tasks are designed to address critical sustainability issues at both the European and global sustainability levels. The assessment of these tasks is facilitated by the *dim²ension* system (the *technical project outcome*), a newly developed digital tool. The dissemination of the *educational outcomes* of the *dim²ension* project is an integral part of the *educational results* of the project. To this end, in-service teacher training and university seminars for pre-service teachers are designed and carried out within the project. The overarching objective of this dissemination is twofold: firstly, to ensure the effective communication of the technical and material outcomes of the project, and secondly, to facilitate the uptake and use of the project material and technology in everyday practices in schools. Finally, to ensure high quality, the *dim²ension* project includes *research-based project results*: These include the development of this document, i.e., the theoretical framework of the project. In addition, intermediate development steps and final project results are evaluated from a scientific point of view.

To reiterate the fundamental idea underpinning the *dim²ension* project: the objective is to provide support for mathematical modelling in the context of sustainability with a new digital tool. Consequently, the project offers innovative approaches to the teaching and learning of, and through, mathematical modelling. Within a highly motivating digital learning environment, students have the opportunity to engage with modelling tasks that address pressing issues in their lives from both a local and global perspective. The contributions of the *dim²ension* project to the field of mathematics education in general, and mathematical modelling in particular, are threefold, encompassing technical, material and educational innovations, thereby underscoring the fundamental principle of mathematical modelling: *the investigation of the real world through the lens of mathematics*!

1.2 Outline of the paper

This section provides an overview of the structure of the theoretical white paper. The document is divided into several chapters which in turn have multiple sections, each addressing key aspects of the *dim²ension* project. Chapter 2 introduces the concepts of models, mathematical models, and modelling, while Chapter 3 discusses Education for Sustainable Development and its connection to the SDGs. Chapter 4 explores STE(A)M education, and Chapter 5 outlines the characteristics of engaging and productive tasks. Chapter 6 focuses on mobile learning and the use of web portals and apps, and Chapter 7 presents the project's framework for task design and classification. Finally, Chapter 8 offers a conclusion and outlook.

2. Models, mathematical models and modelling

This chapter provides an overview of the key concepts of models, mathematical models, and modelling, as employed in the *dim²ension* project. In accordance with Niss (1996), who emphasise the imperative of lucid and precise definitions of the concepts implicated in scientific work and endeavours, this chapter aims to provide a non-exhaustive introduction and overview of the key concepts of (mathematical) models and (mathematical) modelling adapted and used in the *dim²ension* project.

The chapter is structured in the following way: firstly, the notion of models is discussed more generally, and then the focus shifts to *mathematical* models. Next mathematical modelling from an educational perspective is discussed. The ensuing discussion commence with briefly highlighting the role and aspects of the use of digital tools and technology in mathematical modelling generally, leading up to the end of the chapter, which contain a summary of important considerations for the *dim²ension* project with respect to the teaching and learning of, and with, mathematical modelling, in consideration of the background presented in the chapter.

2.1 Models

The term *model* has a rich etymological history that can be traced back to the Latin word *modulus*, a diminutive form of *modus*, meaning measure, size or standard. This Latin term underwent a linguistic shift, first to Italian as *modello* and subsequently to French as *modèle*, before ultimately entering the English language in the late 16th century. Initially, it was employed to denote a set of architectural plans or designs, however, its subsequent evolution has been substantial (Harper, n.d.; Merriam-Webster, n.d.).

For example, the Oxford Dictionary of English (2024, December) lists five interpretations for the term "model" when looked up as the headword entry (and not including interpretations listed as rare or obsolete):

1. **A three-dimensional representation** of a person, object, or proposed structure, typically smaller than the original; such as an architect creating a scale model of a new building to present to clients.
2. **An example to follow or imitate**, such as a model essay used by a teacher to show students how to structure their writing, an algorithm, or a person setting a good example for how to act and behave in a particular situation.
3. **A simplified description**, often mathematical, of a system or process to aid in calculations and predictions; such as a climate model developed by scientists to predict future weather patterns.

4. **A person employed to display clothing by wearing it or to pose for an artist or photographer**, such as a fashion model walking down the runway showcasing the latest designer collection or a person hired by an artist to pose for a portrait painting.
5. **A specific design or version of a product, machine or device**, such as a new released car model from a car manufacturer, or the latest model of the smartphone from a particular brand.

As illustrated by the meanings and interpretations above, models can be either concrete, such as scaled replicas or illustrations (e.g. interpretations 1, 5), or abstract, like mental constructs or theories (e.g. interpretations 3).

In a mathematical educational context, one general notion of models is presented in the so-called *models and modelling perspective on teaching and learning and problem solving* (Lesh & Doerr, 2003b). In this perspective, "*[m]odels* are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviours of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently" (Lesh & Doerr, 2003a, p. 10, italics in original).

The advantage of such a general conceptualisation of a model is that it facilitates connections between different (scientific) disciplines and theoretical frameworks. Models conceived purely as conceptual systems (without necessarily requiring the model to be expressible using an external notation system) are also referred to as *mental models*, and are used to denote the internal representation that individuals create and use to make sense of sensory and experiential input, data and information (Greca & Moreira, 2000). Models that can be expressed externally and independently of the individual, and thus can be used as a means of communication between individuals, are called *external models* (Jones et al., 2011). Indeed, according Black (1962), and as discussed in Jeppson and Haglund (2003), external models can be categorized as follows:

- **Scale Models**: Enlarging small (e.g., molecules) or large (e.g., solar systems) objects to a manageable size while retaining proportions.
- **Analogue Models**: Created in a different medium but maintaining the structure and relationships of the original phenomenon (e.g. solar systems and the structure of the atom according to Bohr).
- **Mathematical Models**: Rationalizing variables to describe natural phenomena, formulating hypotheses, calculating consequences, and testing them through experiments (e.g., Newton's second law of motion: F=ma).
- **Theoretical Models**: Inviting thought experiments, leading to theoretical advances (e.g., Einstein's elevator thought experiment in the context of special relativity).

Each type of model serves a distinct purpose within different scientific and educational contexts, highlighting the importance of selecting the appropriate model when designing educational frameworks.

Importantly, Black (1962) also highlighted model *archetypes* as systematically coherent ideas in one field that can be transferred by analogy to other fields, often unconsciously. Two such examples pointed out by Jeppson and Haglund (2003) are how physics concepts such as "pressure" or "tension" are used in psychology, and how cognitive science borrows concepts from computer science and computational processes.

Another aspect used to distinguish different types of models, discussed by Jeppson and Haglund (2003) in the context of science and science education, are Gilbert's (2004) *modes of representation*. These modes include:

- **Concrete physical representations**: Three dimensional physical representations of for example scale models; for example, a 3D-printed human heart.
- **Verbal representations**: Descriptions and explanations that convey scientific ideas through language; for example, describe the cycle of water using story telling.
- **Gestural representations**: Physical gestures and movements that can help in understanding and communicating scientific concepts; for example, gesturing the shape of a parabola while explaining quadratic functions.
- **Visual representations**: Diagrams, drawings, and other visual tools that help students visualize complex scientific concepts; for example, use Sankey diagrams to illustrate how flow, movement or change of quantities varies within a complex system.
- **Mathematical or symbolic representations**: Equations and formulas that provide a quantitative understanding of scientific phenomena; for example, using algebra to solve a geometrical problem.

These modes of representation all have parallels in the literature on mathematics education, as evidenced by the work of for example Doerr et al. (2017); Goldin (2020); Lesh (1979), Lesh et al. (1987); and Vergnaud (1998). See Figure 2.1 below of the Lesh translation model as an illustration.

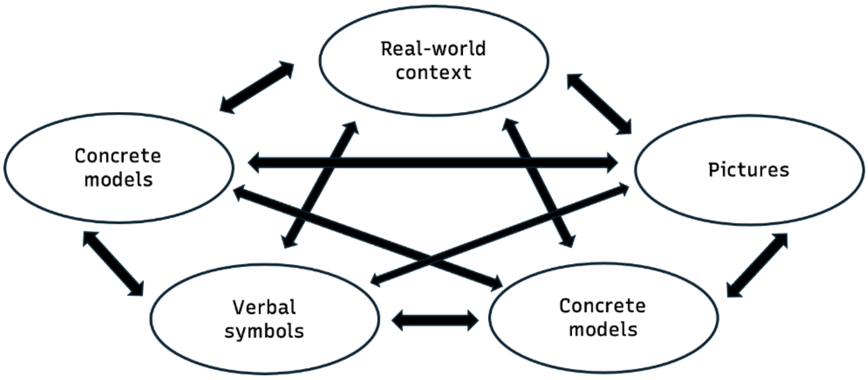

Figure 2.1. *The Lesh translation models* of five representational modes for capturing elementary mathematics ideas (Lesh, 1979), here recreated based on Cramer (2003, p. 449)

It is noteworthy that since the seminal work of Gilbert (2004), a substantial body of literature has emerged exploring novel directions in modes of representation in science and education, particularly those grounded in digital and computational technologies (Storey & Baskerville, 2024; Upmeier zu Belzen et al., 2019). Depending on the perspective adopted, these novel approaches can be regarded as either the introduction of fundamentally new modes of representation or as a means to gain a richer and more nuanced understanding of Gilbert's (2004) *visual-* and *mathematical or symbolic representations*.

2.1.1 Defining models

In the *dim²ension* project, we adopt a broad understanding of the notion of model (c.f. Black (1962); Gilbert (2004)) following Lesh and Doerr (2003b), and define a model to be a conceptual system that is used to create meaning in a given situation for the person(s) using or working with the model.

It should be noted that for our purposes, it is important that it is possible to express and represent this conceptual system using some external representation by the individual working with or using the model. This is essential if the model in question is to become an object of study, discussion and revision.

2.2 Mathematical models

Given a definition and understanding of the notion of model, *mathematical models* are often described as a subset of models expressed using mathematical nomenclature and syntax (Frejd, 2014). Indeed, a straightforward and intuitive understanding of a mathematical model is that it encompasses any of the meanings described above in the list from the Oxford Dictionary of English in section 2.1, except for the sense of (4), which is expressed using mathematical language and syntax.

Within (applied) mathematical disciplines, mathematical models are often classified based on the inherent mathematical properties of the models. According to Lee (2022), such classification schemes typically discuss mathematical models in terms of

- Linear vs. Nonlinear Models:
    - Linear Models: All operations in the mathematical formula are linear. For example, in statistical linear models, the relationship is linear in the parameters but can be irregular in the predictors.
    - Nonlinear Models: At least one operation in the mathematical formula is nonlinear. For instance, a differential equation with nonlinear expressions or a mathematical programming model where one or more constraints are nonlinear.
- Dynamic vs. Static Models:
    - Dynamic Models: These models account for changes in the system's state over time, often using differential or difference equations.
    - Static Models: These models calculate the system in balance and are time-invariant, representing a steady-state condition.
- Explicit vs. Implicit Models:
    - Explicit Models: All input parameters are known, and the outcome variables can be computed using a finite sequence of computations.
    - Implicit Models: Only the output parameters are known, and the corresponding inputs must be solved iteratively, such as using Newton's or Broyden's method.
- Discrete vs. Continuous Models:
    - Discrete Models: Objects are treated as discrete entities, like particles in a molecular model or states in a statistical model.
    - Continuous Models: Objects are represented as continuous, such as the velocity field of fluid in pipe flows, temperatures, stresses in a solid, or an electric field due to a point charge.

Velten (2009) makes a similar distinction between different types of mathematical models structured around a defining triple (S, Q, M): “A mathematical model is a triplet (S, Q, M) where S is a system, Q is a question relating to S, and M is a set of mathematical statements $M = \{S_1, S_2, \ldots, S_n\}$ which can be used to answer Q” (p. 12). A *system* is here understood broadly as the object of interest and study, and may be an aspect of the real word or a more abstract construction or entity. Velten uses the triple (S, Q, M) to classify models with respect to S, Q, and M respectively, and provides the following examples (pp. 40–44), here selectively summarised in Table 2.1 below:

Table 2.1. Examples of Velten’s (2009, pp. 40–44) classification of mathematical models

S – The potential system of interest	**Q – The nature of the question asked given S**	**M – The mathematical structure used to answer Q**
Psychological systems	Speculation	Analytical – numerical
Social systems	Prediction	Continuous - discrete
Economic systems	Analysis	Differential equations
Biological systems	Control	Integral equations
Chemical systems	Design	Linear – nonlinear
Mechanical systems	…	Autonomous – nonautonomous
Electrical circuits		Difference equations
…		Algebraic equations
		…

In mathematics education research, Velten's view is pre-dated and echoed by Niss (1989), and Niss and Blum (2020), who describe the similar triplet (D, f, M). Here, D refers to the extra-mathematical domain corresponding to Velten's (2009) system (S); M represents the mathematical domain providing the mathematical representation of the and is similar to Velten's (2009) mathematical statements (M = {S_1, S_2, …, S_n }); and, f provides the mapping connecting the extra-mathematical domain D and the mathematical domain M - which does not directly correspond to Velten's (2009) notion of questions (Q), but rather f can be seen as the instantiation and operationalisation of such a question.

To illustrate this triplet, we consider a historical example of mathematical modelling: the trajectory of a cannonball, as discussed by Swetz (1989). This example draws on the work of Galileo Galilei in the early seventeenth century, who was among the first to study projectile motion systematically using mathematics. In the early 1600s, Galileo challenged the Aristotelian view that heavier objects fall faster than lighter ones. Through experimentation and thought experiments, he proposed that the path of a projectile under uniform gravity (neglecting air resistance) is parabolic. This insight laid the groundwork for classical mechanics and was pivotal in the development of mathematical modelling in physics. In terms of the triplet (D, f, M), solving the partially oriented modelling problem of how to aim a cannon to hit a particular target can be understood as involving:

- **An extra-mathematical domain, D:** The real-world situation of a cannonball being fired from a cannon with the intent to hit a given target. This includes physical parameters describing how objects behave in nature, such as initial velocity, launch angle, and gravitational acceleration.

- **A mapping f:** A corresponding relation of the idealised important factors and variables in the situation (based on made assumptions, e.g. neglecting air resistance and assuming constant gravity) to the chosen mathematicised counterparts (e.g. time t, horizontal distance x and vertical height y). This mapping translates the physical situation into a mathematical form (and vice versa).

- **A Mathematical domain, M:** Analytic geometry (parabola); with the resulting mathematical model, which in this case is a 2D describes the trajectory of the cannonball as parabolic curve:

$$y(x) = x\, tan\, tan\, (\theta) - \frac{gx^2}{2v_0^2(\theta)} \;,$$

  where x is the horizontal distance, y is the vertical height, θ is the angle of launch, v_0 is the initial velocity, and g is the acceleration due to gravity.

This model allows predictions about the cannonball's path, such as its maximum height, range, and time of flight. It exemplifies how a real-world phenomenon is abstracted into a mathematical structure through modelling.

In terms of the models and modelling perspective (cf. Lesh & Doerr, 2003b), mathematical models are a particular type of conceptual systems, namely those specified by "[a] *mathematical model* focuses on structural characteristics … of the relevant systems" (Lesh & Doerr, 2003a, p. 10, italics in original).

It is worth noting that, in addition to these classifications, mathematical models can be classified from a functional perspective based on their predictive accuracy, adaptability, and the extent to which they simplify complex real-world problems without losing essential information (Czocher, 2018).

2.2.1 Defining mathematical models

In the *dim²ension* project, a mathematical model is defined as a conceptual system that can be externally specified and expressed using mathematical language, nomenclature, syntax, and structure. This definition aligns with the broader understanding of models as tools for creating meaning in a given situation (Lesh & Doerr, 2003a). The model must facilitate meaning-making activities, encompassing various purposes and goals for engaging in mathematical modelling.

In line with the inclusive conceptualisation and applicability of the *dim²ension* project, we define a mathematical model to be a model in the sense defined in Chapter 2.1.1, i.e. a conceptual system that is used to create meaning in a given situation for the person(s) using or working with the model, which can additionally be externally specified and expressed using mathematical language, nomenclature, syntax and structure.

It should be noted that *creating meaning* is to be understood in a broad sense, encompassing different types of meaning-making activities as well as meaning-making for multiple purposes, i.e. ways of working, reasons and goals for engaging in mathematical modelling. Furthermore, our definition aligns well with Niss (1989) and Niss and Blum (2020) in that at a general level of description, a mathematical model according to our adapted understanding in the *dim²ension* project can be expressed in terms of the triplet (D, f, M).

2.3 Mathematical modelling from an educational perspective

As the title of the book edited by Murthy et al. (1990) suggests, mathematical modelling has long been recognised and promoted as a tool for solving problems in engineering, physical, biological and social sciences. Indeed, recent developments in mathematical modelling and applied mathematics are significantly transforming various fields, driven by advances in technology and interdisciplinary collaboration. Key trends include the integration of machine learning and big data analytics (El-Amin, 2025), which facilitates the integration of mathematical modelling with different scientific disciplines, fostering innovative solutions to contemporary challenges (Pinto et al., 2023). Frejd and Bergsten's (2018) study shows that the concept of mathematical modelling has multiple interpretations among professional modellers. These interpretations can be grouped into four main aspects:

- **Description** – How models describe real-world phenomena.
- **Understanding** –The role of models in enhancing comprehension.
- **Abstraction** –The process of simplifying complex systems into manageable models.
- **Negotiation** –The collaborative aspect of developing and refining models.

Similarly, as the terms 'models' and 'modelling' have different meanings in science and in everyday life, so too do the terms 'mathematical models' and 'mathematical modelling' in mathematics education research. Ogborn (1994), quoted in Molyneux-Hodgson et al. (1999, p. 176), describes modelling as “thinking about one thing in terms of simpler artificial things.” In mathematics education research, these simpler artificial things often refer to mathematical vocabulary and syntax. Lingefjärd (2006), drawing on Swetz and Hartzler (1991), defines mathematical modelling as “a mathematical process that involves observing a phenomenon, conjecturing relationships, applying mathematical analyses (equations, symbolic structures, etc.), obtaining mathematical results, and reinterpreting the model" (p. 96). Niss and Blum (2020) on the other describe mathematical modelling as a comprehensive problem-solving approach. According to these authors, this process involves translating real-world problems, situations, or phenomena into mathematical terms, resulting in a model that aids in understanding and addressing the given task or situation. A similar view is expressed by Lesh and Doerr (2003b), where mathematical models and modelling "are used to construct, describe, or explain the behaviours of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently" (p. 10).

There are many approaches and perspectives on mathematical modelling in mathematics education research (Ärlebäck et al., 2023; Blum et al., 2007; Siller et al., 2024b). Sriraman and Kaiser (2006) highlight the lack of a homogenous understanding of modelling and its epistemological backgrounds within international discussions, calling for clearer definitions to improve communication and discussions. Hamson (2003) also notes the potential for misuse and misunderstanding of the term modelling, emphasizing the need for clarity in mathematics classrooms.

Although the notions of mathematical models and modelling are used ambiguously, Kaiser et al. (2006) note that research in mathematics education often includes a general description of the modelling process. One common description is the modelling cycle, which schematically connects the extra-mathematical and mathematical domains (Niss et al., 2007; Niss & Blum, 2020). For example, Borromeo Ferri (2006) outlines the modelling process in six phases: real situation, mental representation, real model, mathematical model, mathematical result, and real results, with transitions such as understanding, simplifying, mathematizing, working mathematically, interpreting, and validating. These cycles may vary depending on the research focus, highlighting different aspects of the modelling process (Haines & Crouch, 2010; Jablonka, 1996).

The conceptualisation of mathematical modelling as a six-phased process with given transitions (c.f. Borromeo Ferri, 2006), is a simplified and idealised representation of the process and has been criticised. First, the basis for such a cyclical conceptualisation of the modelling process is a dichotomous division between the intra- and the extra-mathematical worlds or domains which can be seen as artificial, raising questions about its foundation and consequences. Second, empirical findings suggest that the modelling process is not cyclical; modellers often jump between stages in a non-cyclical, unsystematic manner (Ärlebäck & Bergsten, 2010; Borromeo Ferri, 2007a; 2007b). To capture this 'stochastic' nature, Voskoglou (2007) uses a finite Markov chain to describe the modelling process in classrooms. In addition, the transitions between steps are not unidirectional but dialectical. Skov Hansen et al (1999) illustrate this with a network that connects phases without specifying directions, taking into account the stochastic character. Third, the transition from the real model to the mathematical model, termed mathematisation, has been questioned by Jablonka and Gellert (2007). They argue that the distinction between the real world and mathematics is untenable, especially in classrooms, as the 'real model' is influenced by available mathematics. They conclude that "Mathematisation within the circular process of mathematical modelling is – epistemologically regarded – a potentially misleading construct and it is – pedagogically – of debatable value" (pp. 5-6). This aspect has also been addressed and discussed in terms of so-called *implemented anticipation* (Niss, 2010; Stillman & Brown, 2012; 2014).

The other common framework of mathematical modelling involves modelling competence, competency, or competencies. However, definitions of these terms vary in the literature (Blomhøj & Højgaard Jensen, 2007; Blum et al, 2007; Cevikbas et al., 2022; Niss & Blum, 2020; Maaß, 2006). Blomhøj and Højgaard (2007) note that 'competence' has become a buzzword in mathematics education, adding flavor to discussions or planning by its mere mention (p. 45). Greer and Verschaffel (2007) differentiate between implicit, explicit, and critical modelling competencies. Implicit competencies involve unrecognized modelling activities, explicit competencies focus on the modelling process, and critical competencies involve reflecting on the roles and consequences of modelling in various contexts.

The definition of modelling competency by Blomhøj and Højgaard Jensen (2003; 2007) originates from the Danish KOM-project (Niss & Jensen, 2002; Niss, 2003). This definition offers a holistic view of mastering mathematical modelling and provides a framework to describe, support, and assess student progress along three dimensions: degree of coverage, technical level, and radius of action. Degree of coverage focuses on the parts of modelling students work on and their reflective abilities; technical level reflects the mathematics used and flexibility in its application; and radius of action accounts for the contexts and domains in which students can perform mathematical modelling.

Henning and Keune (2007) propose a three-level framework for mathematical modelling: Recognition and understanding of modelling (1st level), Independent modelling (2nd level), and Meta-reflection on modelling (3rd level) (p. 227). Critical modelling or meta-reflection relates to model validation, which extends beyond the mathematical link to include extra-mathematical knowledge and societal values. Jablonka (1996) examined the epistemological status of mathematical models in teaching materials, identifying validation as a problematic issue often avoided or trivialized. Jablonka and Gellert (2007) describe mathematisation and demathematisation as social processes, with mathematics generating new realities by describing, colonizing, and transforming them (p. 6).

Despite the diverse definitions and notions of mathematical models and modelling, Kaiser et al. (2006) expressed optimism about understanding the different approaches and their connections. They argue that there is already "a global theory for teaching and learning mathematical modelling, in the sense of a system of connected viewpoints covering all didactical levels" (p. 82), though this theory is "far from being complete" (p. 82).

In the last 20 years, efforts have aimed to clarify and differentiate approaches to mathematical models and modelling (Kaiser & Sriraman, 2006; Kaiser et al., 2007), a discussion that is on-going (Ärlebäck et al., accepted). According to Kaiser and Sriraman (2006), perspectives on mathematical modelling can be classified as realistic or applied modelling, contextual modelling, educational modelling (didactical or conceptual focus), socio-critical modelling, epistemological or theoretical modelling, or cognitive modelling. For further background and discussion, see Kaiser and Sriraman (2006); Kaiser et al. (2006); Sriraman et al. (2006); Kaiser et al. (2007); Ärlebäck et al., 2023).

2.3.1 Defining mathematical modelling

In the *dim²ension* project, we define a *mathematical modelling* to broadly be aimed at investigating real-world phenomena using mathematics (Blum, 2015). It is the complex process of applying mathematical knowledge to describe, analyse or evaluate real-world situations (Blum & Niss, 1991) using mathematical models as defined in subsection 2.2.1.

2.4 Mathematical modelling with digital tools

Hansson (2020) argues that the connection between technology and mathematics can be traced back to the dawn of civilization, yet this is an area that has not received much attention in research. This connection is described as inherently dynamic and evolving over time. It has influenced areas ranging from "counting and arithmetic, via the use of mathematics in weaving, building and other trades, and the introduction of calculus to solve technological problems, to the modern use of computers to solve both technological and mathematical problems" (p. 117). However, a general conclusion is that the development of mathematical knowledge depends on the available and employed technology, that the definition of the concept of a (technological) computation has changed and evolved, and that the relationship between the usefulness of mathematics in technology and vice versa is symbiotic (Hansson, 2020).

Over the past three decades in particular, the integration of digital tools into mathematical modelling has significantly transformed both the practice of mathematics and its applications across disciplines. This transformation is not merely technological but epistemological, reshaping how mathematical knowledge is constructed, validated, and applied.

For example, the introduction of symbolic computation platforms such as Mathematica and Maple in the 1990s marked a turning point in mathematical research. These tools enabled the automation of algebraic manipulations and symbolic integration, allowing researchers to focus on higher-level conceptual work. Numerical computing environments like MATLAB and Python's SciPy ecosystem further expanded the toolkit, making it easier to simulate complex systems and solve large-scale numerical problems (Huang et al., 2025).

Another example in pure mathematics discussed by (Avigad, 2020), is the rise of formal proof verification systems such as Coq, Lean, and Isabelle has introduced a new paradigm of rigor. These tools allow for the formalization and machine verification of mathematical proofs, which has led to the resolution of long-standing conjectures and the development of new formal libraries of mathematics. This shift has sparked philosophical discussions about the nature of proof and the role of human intuition in mathematical discovery.

Relatedly, the emergence of machine learning and artificial intelligence has further expanded the boundaries of mathematical modelling. Physics-informed neural networks (PINNs), for example, use deep learning to approximate solutions to partial differential equations by embedding physical laws into the training process. This approach has been applied successfully in fluid dynamics, materials science, and biomedical engineering (Raissi et al., 2019). In addition, more recently large language models (LLMs) have been evaluated for their ability to construct and solve mathematical models (Huang et al., 2025).

Digital tools have also transformed the application of mathematics in other scientific domains. Zienkiewicz et al. (2013) discusses how in engineering and physics, finite element methods (FEM) implemented in platforms like COMSOL Multiphysics and ANSYS have become standard for simulating physical systems. These tools allow for the modelling of stress, heat transfer, and electromagnetic fields with high fidelity, often incorporating real-world data and boundary conditions.

In the life sciences, mathematical models of biological systems—ranging from gene regulatory networks to epidemiological dynamics—have become more sophisticated through the use of agent-based modelling and stochastic simulations (Wilensky & Rand, 2015). These models are often implemented using platforms like NetLogo or custom Python environments, enabling researchers to simulate complex interactions and emergent behaviours. Economics and the social sciences have similarly benefited from digital modelling tools. Agent-based models, game theory simulations, and network analysis have become more accessible and powerful, allowing for the exploration of market dynamics, social influence, and policy impacts in silico. These developments have been supported by the increasing availability of computational resources and open-source libraries (Wilensky & Rand, 2015).

The epistemological implications of these changes are profound. As digital tools become more integral to mathematical practice, they are not merely augmenting traditional methods—they are reshaping what it means to do mathematics. The boundary between theoretical and applied mathematics is becoming increasingly porous, with computational experimentation and simulation playing a central role in both domains.

2.5 Teaching and learning (with) mathematical modelling

This section focuses on teaching and learning of, and through, mathematical modelling. First, it elaborates on mathematical modelling as an area of educational research. Then, in a section highly relevant to the *dim²ension* project, teaching and learning mathematical modelling using digital tools is discussed.

2.5.1 Mathematical modelling in educational research

Mathematical modelling, as previously discussed, serves as a bridge between abstract mathematical theory and real-world application. In educational contexts, this bridge is not merely a metaphor but a pedagogical imperative. Teaching and learning (with) mathematical modelling involves engaging students in the process of constructing, interpreting, and refining mathematical representations of real-world phenomena. This process fosters a deeper understanding of mathematics while simultaneously cultivating critical thinking, creativity, and interdisciplinary competencies (Blum & Niss, 1991; Lesh & Doerr, 2003a; Niss & Blum, 2020).

From a didactical perspective, modelling is not only a method for applying mathematics but also a means of learning mathematics. The models and modelling perspective (Lesh & Doerr, 2003a) emphasizes that models are conceptual systems used to describe, explain, and predict the behaviour of other systems. In this view, modelling becomes a central activity in mathematics education, enabling students to engage in authentic problem-solving and to experience mathematics as a meaningful and dynamic discipline.

The educational potential of mathematical modelling is multifaceted. It supports the development of modelling competencies (Blomhøj & Jensen, 2003; Maaß, 2006), which include the ability to understand real-world problems, translate them into mathematical terms, solve the resulting mathematical problems, and interpret and validate the results. These competencies are essential for navigating complex societal challenges and are closely aligned with the goals of Education for Sustainable Development (Siller et al., 2024b).

However, the implementation of modelling in classroom practice presents significant challenges. Teachers often struggle with designing appropriate modelling tasks, diagnosing students' difficulties, and assessing modelling processes (Borromeo Ferri & Blum, 2010; Wess et al., 2021). Students, in turn, face difficulties in dealing with open-ended tasks, identifying suitable models, and validating their solutions (Blum, 2015; Galbraith & Stillman, 2006). These challenges underscore the need for robust pedagogical frameworks and supportive learning environments.

Recent research highlights the transformative potential of digital tools in addressing these challenges. Digital technologies can support modelling activities by providing dynamic representations, facilitating data collection and analysis, and enabling simulations of complex systems (Cevikbas et al., 2023; Siller et al., 2023a). For example, the integration of computer algebra systems (CAS), dynamic geometry software (DGS), and spreadsheets allows for the exploration of mathematical models from multiple perspectives—symbolic, graphical, and numerical—thereby enriching students' understanding and engagement (Dörfler & Blum, 1989; Greefrath & Siller, 2018).

Moreover, the use of digital tools reconfigures the roles of teachers and students. Teachers become facilitators and coaches, guiding students through modelling processes rather than delivering fixed content. Students, in turn, are empowered to take greater responsibility for their learning, exploring modelling scenarios independently or collaboratively within technology-rich environments (Geiger, 2011; Siller & Greefrath, 2010).

Nevertheless, the use of technology in modelling education must be reflective and purposeful. If digital tools are used merely to automate calculations without fostering conceptual understanding or critical reflection, their pedagogical value is diminished. As such, the integration of technology should be guided by clear educational goals and embedded within well-designed modelling tasks that promote inquiry, reasoning, and communication (Doerr & Pratt, 2008; Schaap et al., 2011).

In summary, teaching and learning (with) mathematical modelling is a rich and evolving field of educational research. It encompasses theoretical, pedagogical, and technological dimensions, all of which must be carefully considered to realize the full potential of modelling in mathematics education. The *dim²ension* project builds on this foundation by developing digital tools and task frameworks that support the implementation of modelling activities in diverse educational contexts, with a particular focus on sustainability and interdisciplinary learning.

2.5.2 Mathematical modelling using digital resources

The utilisation of digital resources in the context of teaching, particularly with regard to modelling, has been extensively explored in numerous volumes of the 'International Congress on Mathematical Education' 13 (ICME 13) (e.g. Ball et al., 2018; Hegedus et al., 2017). The integration of digital media with modelling activities gives rise to a diverse array of modelling tasks (Drijvers et al., 2016). On the one hand, there are open modelling projects in which students are given an authentic, open, real-life problem in a technology-rich environment that they can explore, solve and present (Geiger & Redmond, 2013). Conversely, there are tasks that do not inherently fall under the purview of modelling, yet they offer opportunities for digital media integration. These tasks encompass structured scenarios, pre-established mathematical models and artificial problem instances, where learners employ digital media to find solutions (Drijvers et al., 2016). The wide spectrum between these two extremes makes mathematical modelling and digital media a fascinating field of study.

The integration of digital media in mathematics education to address real-life problems gives rise to novel opportunities for content development. For instance, contentious topics within the curriculum that for various reasons – including the necessity for huge amounts of data or complicated calculations – were previously difficult or impossible to address in the classroom can now be tackled more easily. Consequently, new avenues for exploring mathematical situations are opened up (Drijvers, 2003). A pedagogical approach that integrates digital media in a productive way is poised to provide substantial support to both teachers and students, particularly in the context of grappling with authentic modelling scenarios.

The different functions of digital tools in mathematics education come into play at various points in real-world problems. Some possibilities for using digital tools in a modelling process are illustrated in the seven-step modelling cycle according to Blum and Leiß (2005). At the beginning of the modelling process, digital tools might be used to conduct research on the topic. After mathematical work with the model, they might rather be used to control and check mathematical results. Calculations take place in the modelling cycle between mathematical model and mathematical results. Some authors focus particularly on calculation when using digital mathematics tools (e.g. Pierce, 2005; Doerr and Pratt, 2008) or at least on the area of the modelling cycle between the real model and mathematical results (Galbraith and Stillman, 2006, p. 147). If we take a closer look at this step of mathematical work with digital tools, further aspects arise. Using digital tools for calculations in the mathematical model requires the translation of the mathematical model into a model for the digital tool. This digital tool model then provides results that in turn have to be transferred into the mathematical results. This focus on calculating shows that the use of digital tools in modelling requires further translations and models.

Schaap et al. (2011, p. 144) see the opportunities of digital tools particularly in understanding the modelling problem. Here, the simplification achieved by drawing a picture of the situation and mathematicising it with the help of concrete examples is particularly emphasised. However, they also point out the potential for validation. Other authors particularly emphasise the possibilities offered by digital tools through various forms of presentation (Confrey and Maloney 2007; Doerr and Pratt 2008, p. 265 ff.). This can also be seen as an indication of the influence of the digital tool used on the mathematical model. In accordance with the findings of Gellert, Jablonka and Keitel (2001), we suggest that the utilisation of digital media in mathematical modelling not only impacts the mathematical knowledge of students but also exerts a significant influence on the mathematical models employed.

Geiger (2011, p. 312) and Daher and Shahbari (2015) see the use of digital tools at different points, depending on the people examined. Thus, a circular model with tool use integrated at many points can be established. These considerations make it clear that the use of digital tools can be important in all phases (Greefrath et al. 2011). Recent studies show that this integrated view describes the actual modelling activities with digital tools better than an extended modelling cycle, which emphasises tool use at one point (see Greefrath and Siller, 2018).

With regard to students working in didactically pre-structured learning environments, the step ‘syntax’ (transition mathematics-technology, MT) must be considered from two perspectives: It can involve the implementation of mathematical structures in computer language, which requires semantic and syntactic knowledge on the part of the implementer. In simple cases, such as when using an GTR, this step can be taken by the learners themselves.

More elaborate digital models, such as (prepared) dynamic worksheets in DGS, pre-structured tables in TPK, animations or even virtual experimental environments or simulations, can be prepared from a didactic point of view and then used by learners; to do so, they use sliders, checkboxes, input fields or other interactive elements of the respective technology-based implementation. This also requires users to have semantic knowledge of the respective application (operational question: what happens if I pull the slider?), but this can usually be acquired directly through trial and error. Syntactic requirements, on the other hand, take a back seat. In this second case, it is not about implementing the model, but rather about discovering and exploiting the meaningful use of the respective digital technology.

Furthermore, the model of Siller and Greefrath (2010) seems to assume that, starting from the real situation, the mathematical description of the problem is first carried out and then, on this basis, the virtual description – this sequence is also found in Bossel (1992) and Hoffmann and Witterstein (2014, p. 2). This may describe working with real-life modelling problems that have not been didactically reduced or pre-reduced. However, when technologies are used as aids in learning mathematics, their use is often prepared and pre-structured by teachers, experts or mentors. From the learners' point of view, this approach smoothes the direct transition from the real situation to the digital world. This makes it possible to use digital technologies to explore the mathematical model – without assuming that it is known.

The above discussions shows that modelling activities can, and are, often supported by the use of digital tools. For example, Bossel (1992, p. 13) refers to ‘computer simulation’ (Fig. 6), Eck et al. (2008, p. VII) highlight ‘numerical methods [...] in the actual processing of technical and scientific problems’ using technological tools and Hoffmann and Witterstein (2014, p. 2) use the term ‘numerical simulation’. Blum and Leiß (2005) also consider the use of software in the form of simulations in their approach, at least implicitly.

Siller and Greefrath (2010) make the role of technology use in mathematical modelling explicit by defining a third world, the ‘technological world’, between the two worlds of ‘reality’ or ‘the rest of the world’ and ‘mathematics’ described by Pollak (1979) and Blum (1985; 2006) – they distinguish a third world, the ‘technological world’, and highlight two further transitions between these worlds (see Figure 2.2). For example, in order to solve a mathematical modelling problem using technology, the mathematical model must be translated into computer language (‘syntax’, Siller & Greefrath, 2010), and the results of the computer calculation must be transferred back into mathematical language (‘interpretation’, Siller & Greefrath, 2010). The syntax can take different forms such as geometrical construction in GeoGebra or different types of computer code.

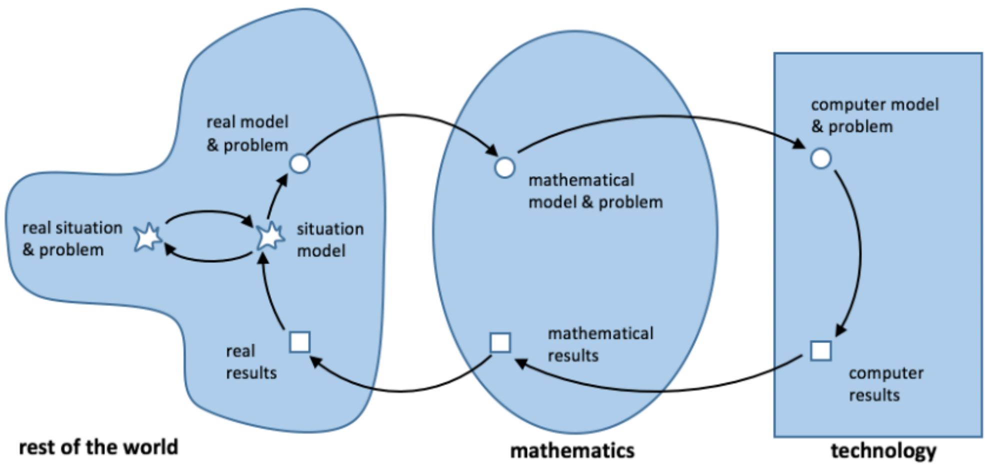

Figure 2.2. The role of technology use in mathematical modelling made explicit by defining a third ‘technological world’ (Siller & Greefrath, 2010)

Taking a broader view, other researchers (Daher & Shahbari, 2015; Galbraith et al., 2003; Geiger et al., 2003; Greefrath et al., 2011) see the use of digital resources as relevant throughout the modelling process, that is, at each stage and at each transitions between stages. This can be seen in the case of the phenomenon of coin migration described by Greefrath and Siller (2018). This perspective is visualized in Figure 2.3.

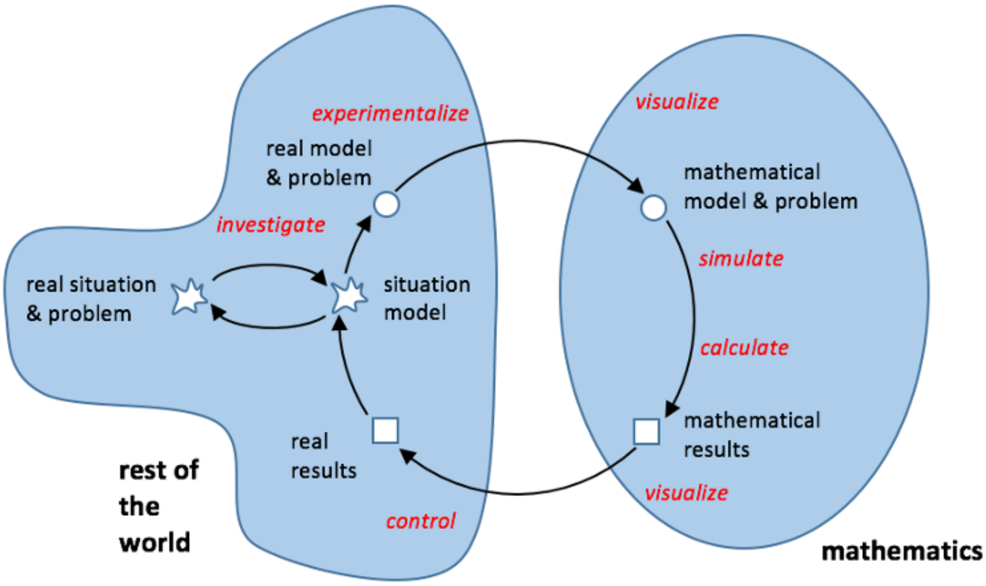

Figure 2.3. The role of technology use in mathematical modelling throughout the different stages of the modelling cycle (Greefrath et al., 2018)

However, when technologies are used as aids in learning mathematics, their use is often prepared and pre-structured by teachers, experts or mentors. We argue, in line with Siller et al. (2023a), that digital resources can be used throughout the entire modelling process, as they can help students gain insights into the essential features of a problem. Thereby, digital resources, both mathematics-specific and non-mathematics-specific (i.e. focused on aspects of communication and information retrieval), can improve teaching and learning in the area of mathematical modelling, by (1) enhancing mathematical skills through the use of digital resources that are specific to mathematics, (2) that are not specific to mathematics, and (3) access to non-mathematical knowledge about communication and information retrieval and about (4) simulated, virtual and augmented worlds (Siller et al., 2023a).

3. Education for Sustainable Development

This chapter begins with a brief background and discussion of the United Nations' (2015) Sustainable Development Goals (SDGs), their connection and relevance to the discussion in the EU and its counterpart, as well as the educational needs and challenges they imply (Chapter 3.1).

In addition, the chapter provides a brief overview of existing research focusing on the challenges and opportunities for teaching and learning sustainable development. Importantly, the chapter also includes a discussion of the learning objectives for the SDGs, which can be divided into the cognitive, socio-emotional and behavioural domains (UNESCO, 2017). These learning objectives are taken up again in one dimension of the *dim²ension* task classification framework. Finally, the importance, role and function of models and modelling in the context of Education for Sustainable Development and in addressing the SDGs are discussed in the final section of the chapter (Chapter 3.2).

3.1 Sustainable Development Goals

In the contemporary era, a plethora of global challenges have been brought to the fore, including but not limited to extreme weather conditions, armed conflicts, diseases, hunger, and economic crises. Consequently, the necessity for a development that 'meets the needs of the present without risking that future generations cannot meet their own needs' (Brundtland, 1987, p. 41) has become paramount.

In 2015, global stakeholders reached a consensus on 17 Sustainable Development Goals (SDGs) (United Nations, 2015), which underscore critical global challenges across the social, environmental and economic domains. UN member states worldwide have committed to the following by 2030: the eradication of poverty, the assurance of freedom and peace, the protection of our planet, and the promotion of inclusive growth (United Nations, 2015); see Figure 3.1 and Table 3.1.

Figure 3.1. United Nations' (2015) 17 Sustainable Development Goals (SDGs). Source: United Nations Communications Materials, https://www.un.org/sustainabledevelopment, Note: *The content of this publication has not been approved by the United Nations and does not reflect the views of the United Nations or its officials or Member States.*

The European Union has adopted these goals and has declared sustainable development to be a 'core principle of the European Union' and an overarching policy priority (European Union, 2024a).The European Commission has established seven key priorities to work towards, including sustainable prosperity and competitiveness, maintaining quality of life by ensuring food security, water and nature, protecting democracy and promoting social justice (European Union, 2024b).

However, a review of current data indicates that the EU has made only limited progress towards the majority of the SDGs (European Union, 2024a). Similarly, the latest global SDG reports highlight that progress towards the 17 agreed goals remains inadequate (Guterres, 2024). This underscores the urgent need for novel approaches to promote sustainable development, with Education for Sustainable Development (ESD) playing a pivotal role in shaping a sustainable future.

Table 3.1. United Nations (2015) 17 Sustainable Development Goals (SDGs)

Goal	Official Name	Description
1	No Poverty	End poverty in all its forms everywhere by ensuring social protection, equal rights to resources, and resilience to economic, social, and environmental shocks.
2	Zero Hunger	End hunger, achieve food security, improve nutrition, and promote sustainable agriculture to ensure everyone has access to sufficient and nutritious food.
3	Good Health and Well-being	Ensure healthy lives and promote well-being for all at all ages by improving healthcare access, reducing mortality, and combating diseases.
4	Quality Education	Ensure inclusive and equitable quality education and promote lifelong learning opportunities for all, from early childhood to adult education.
5	Gender Equality	Achieve gender equality and empower all women and girls by eliminating discrimination, violence, and harmful practices.
6	Clean Water and Sanitation	Ensure availability and sustainable management of water and sanitation for all, focusing on access, quality, and efficiency.
7	Affordable and Clean Energy	Ensure access to affordable, reliable, sustainable, and modern energy for all, especially through renewable sources.
8	Decent Work and Economic Growth	Promote sustained, inclusive, and sustainable economic growth, full and productive employment, and decent work for all.
9	Industry, Innovation and Infrastructure	Build resilient infrastructure, promote inclusive and sustainable industrialization, and foster innovation to drive development.
10	Reduced Inequalities	Reduce inequality within and among countries by addressing income disparities and promoting social, economic, and political inclusion.
11	Sustainable Cities and Communities	Make cities and human settlements inclusive, safe, resilient, and sustainable through better planning and infrastructure.
12	Responsible Consumption and Production	Ensure sustainable consumption and production patterns by reducing waste and promoting resource efficiency.
13	Climate Action	Take urgent action to combat climate change and its impacts by reducing emissions and strengthening resilience.
14	Life Below Water	Conserve and sustainably use the oceans, seas, and marine resources for sustainable development.
15	Life on Land	Protect, restore, and promote sustainable use of terrestrial ecosystems, manage forests, combat desertification, and halt biodiversity loss.
16	Peace, Justice and Strong Institutions	Promote peaceful and inclusive societies, provide access to justice for all, and build effective, accountable institutions.
17	Partnerships for the Goals	Strengthen the means of implementation and revitalize the global partnership for sustainable development through cooperation and resource mobilization.

3.1.1 Education for Sustainable Development

In the context of educational responses to disruptive events, the Education for Sustainable Development (ESD) programme has been identified as a significant initiative by UNESCO (2015, 2017) The overarching objective of the programme is to cultivate awareness among individuals of all age groups concerning pertinent issues, thereby equipping them with the capacity to make informed decisions and to engage proactively in personal and collective endeavours that foster social transformation and environmental stewardship (UNESCO, 2017).

The central goal of Education for Sustainable Development (ESD) is to empower learners to address current, interconnected and complex global challenges by equipping them with the necessary knowledge, skills, values and action competencies (UNESCO, 2017). As a prerequisite for achieving this, learners should be able to recognise, evaluate and act in a complex world (Siller et al., 2025a). Achieving this objective also necessitates integrating various subject areas that reflect our social, ecological and economic interactions.

Mathematics plays a pivotal role, acting as a tool and mediator between different subject areas and providing a foundation for numerous models and decision-making processes (Hopkinson & James, 2010; Lafuente-Lechuga et al., 2020; Skovsmose, 2023; Xenofontos et al., 2021). However, using mathematical methods only in obvious situations, such as data analysis, is insufficient. It is also important to identify key mathematical concepts that may not be immediately obvious in a particular context, such as Education for Sustainable Development (ESD). Furthermore, it is crucial to avoid the unreflective use of mathematics as a mere tool, as this can result in misunderstandings or the overlooking of the limitations of mathematical models. Conversely, reflective application can facilitate a more profound understanding of both the ESD context and the mathematical content and concepts (e.g. Ningsih & Juandi, 2019; Siller et al., 2024b). Therefore, the role of mathematics in ESD must be highlighted and its importance emphasised.

Through engaging in comparative analyses of different mathematical arguments, students develop the ability to critically evaluate situations (Siller et al., 2024b). This process involves identifying potential mathematical methodologies for solving problems, while also recognising the importance of knowledge from other academic disciplines and everyday experiences. Consequently, students are empowered to make informed, evidence-based decisions regarding sustainability issues, manifesting in a propensity and readiness to act that extends beyond the classroom. Combining knowledge and skills in this way fosters active engagement in social discourse, enabling students to articulate well-founded arguments backed by mathematical and statistical evidence. In the long term, this contributes to a transformation in the culture of discussion and is a fundamental aspect of a democratic society. Engaging with ESD topics through critical mathematical reflection in the classroom has been shown to raise students' awareness of the complexity and urgency of sustainability issues (Li & Tsai, 2022; Uitto & Saloranta, 2017). These insights can act as a catalyst for behavioural change, leading to active advocacy for social and political transformation.

Implementing ESD in the classroom requires the creation of robust learning environments and appropriate individual tasks (e.g. Just et al., 2023). Empirical research is essential to demonstrate the effectiveness of ESD, including the identification of the so-called ESD core competencies (Siller et al., 2025b) and the cognitive activation potential of tasks (see Siller et al., 2024a). Such research is vital in highlighting the advantages of integrating ESD into mathematics education. Furthermore, it can counteract the tendency to take action in this subject area and integrate students' concerns (as expressed in the Fridays for Future protests, for example) into everyday teaching.

3.1.2 Objectives of ESD

A central aim of ESD is to empower learners to promote social change and environmental responsibility by addressing complex global challenges head on. In order to achieve this overarching goal, students require the knowledge, skills, values and attitudes that will empower them to contribute to sustainable development (UNESCO, 2017, p. 7). This highlights the importance of promoting a variety of learning objectives when implementing ESD in the classroom, covering the cognitive, socio-emotional, and behavioural domains (UNESCO, 2017).

- **Cognitive domain:** Students need to acquire the knowledge and thinking skills required to understand the SDGs and the challenges involved in achieving them (UNESCO, 2017). This encompasses a wide range of topics that extend beyond purely technical skills (KMK, 2024).
- **Socio-emotional domain:** This domain encompasses the social skills necessary for learners to collaborate, negotiate and communicate effectively in pursuit of the SDGs, as well as the self-reflection skills, values, attitudes and motivations that are crucial for personal development. Students need to be able to reflect critically, recognise and evaluate diverse values, and develop their identity based on different value-based perspectives. Social-emotional learning is essential for fostering solidarity and shared responsibility towards people and the environment (UNESCO, 2017; KMK, 2024).
- **Behavioural domain:** This domain encompasses the competencies required for action, which are central to the field of global development. These competencies include the ability to resolve conflicts and demonstrate understanding and tolerance of ambiguity, as well as creativity and a willingness to innovate. Above all, they involve aligning one's behaviour with the personal principles of a sustainable lifestyle (UNESCO, 2017; KMK, 2024).

These objectives are relevant to all SDGs. It is crucial that ESD goes beyond acquiring knowledge and cognitive skills to address values and attitudes too. Conversely, knowledge and thinking skills are essential for promoting well-informed, fact-based opinions, attitudes, decisions and actions in our XXI century students.

3.2 Models and modelling in the context of the SDGs

Models and modelling techniques are indispensable tools in the pursuit of the United Nations' 2030 Agenda and its 17 Sustainable Development Goals (SDGs). They serve as conceptual and computational frameworks that allow researchers, policymakers, and educators to simulate complex systems, anticipate the consequences of policy decisions, and monitor progress toward sustainability targets. In the context of sustainable development, models help bridge the gap between abstract goals and actionable strategies by offering structured ways to explore interdependencies among economic, environmental, and social systems (Zhang et al., 2016).

Models are particularly valuable in identifying synergies and trade-offs between SDGs. For example, a systems approach enables the mapping of interactions between goals such as climate action (SDG 13), clean water and sanitation (SDG 6), and life on land (SDG 15), revealing leverage points where interventions can yield multiple benefits (Barbier & Burgess, 2017). This holistic perspective is essential for designing integrated policies that avoid unintended consequences and maximize positive outcomes across sectors.

In policy-making, models function as decision-support tools that inform the design, implementation, and evaluation of sustainability strategies. They allow stakeholders to test scenarios, assess risks, and explore the long-term impacts of interventions. For instance, environmental and complex systems modelling has been used to design Earth observation systems that support biodiversity conservation, directly contributing to SDG 15 (Reid et al., 2019). Similarly, macroeconomic models are employed to evaluate the implications of fiscal and trade policies on poverty reduction (SDG 1) and economic growth (SDG 8) (Kumar et al., 2018).

However, the utility of models extends beyond technical analysis. When embedded in participatory processes, modelling becomes a means of fostering dialogue among stakeholders, enhancing transparency, and building trust. Iterative engagement with decision-makers and communities ensures that models are grounded in local realities and that their outputs are both relevant and actionable (Barke et al., 2023).

Despite their potential, current modelling capabilities face significant limitations. Many models still rely heavily on economic indicators such as GDP, which are increasingly criticized for failing to capture the full spectrum of human and environmental well-being (Sun et al., 2024). There is a growing consensus that models must evolve to incorporate broader measures of sustainability, including social equity, ecological resilience, and cultural values (Zaidan & El Fadel, 2024).

Moreover, the complexity of the SDG framework—with its 17 goals and 169 targets—poses a formidable challenge for model integration. No single model can comprehensively address all goals simultaneously, and even the most ambitious modelling pathways often fall short of providing insights into certain targets (Huan et al., 2021). This underscores the need for modular, interoperable modelling approaches that can be tailored to specific contexts and objectives.

Ethical considerations also arise in the use of models, particularly regarding the reductionist nature of indicators and the risk of oversimplifying contested concepts. While indicators are essential for tracking progress, they must be used with caution and complemented by qualitative assessments to ensure a nuanced understanding of development dynamics (Weber & Weber, 2020).

Turning to the realm of education, models and modelling offer powerful pedagogical opportunities for advancing Education for Sustainable Development (ESD). They enable learners to engage with real-world problems, develop systems thinking, and cultivate competencies such as critical thinking, collaboration, and ethical reasoning. Fermi problems, for example, are a form of estimation-based modelling that encourages students to make assumptions, justify their reasoning, and reflect on the implications of their solutions—skills that are directly aligned with the SDGs (Albarracín & Ärlebäck, 2019)

Educational research highlights the importance of integrating modelling into curricula at all levels, from primary education to higher education and teacher training. Experiential learning projects, such as those involving renewable energy or water conservation, provide authentic contexts for applying modelling techniques and understanding sustainability challenges (Monzó-Martínez et al., 2024). However, gaps remain in the representation of SDGs within educational programs, particularly in subjects like mathematics, where modelling is often underutilized (Do & Thinh, 2025).

To address these gaps, there is a need for innovative, cross-curricular pedagogies that embed modelling within broader sustainability narratives. Teacher education programs must also prioritize modelling as a core competency, equipping educators with the tools and confidence to facilitate modelling activities in diverse classroom settings (Bernárdez-Gómez et al., 2024).

In conclusion and to summarise, models and modelling are not merely technical instruments but are central to the transformative vision of the SDGs. They provide the analytical backbone for sustainable development planning, the pedagogical scaffolding for sustainability education, and the communicative bridge between science, policy, and society. To fully realize their potential, modelling practices must become more inclusive, interdisciplinary, and ethically grounded. This entails embracing uncertainty, valuing diverse knowledge systems, and fostering collaborative learning environments where models are co-created and critically examined. However, research focusing on the educational aspects of the SDGs is underdeveloped. Different studies have identified areas in need of more systematic research, such as: the limited SDG integration across subjects (Torres, 2021); inconsistent curriculum representation (Monzó-Martínez et al., 2024); insufficient teacher training in ESD generally (Gómez-Gómez & García-Lázaro, 2023); Bernárdez-Gómez et al., 2024) ; the lack of dedicated ESD courses in teacher education in particular (Edwards et al., 2020); institutional barriers in higher education (Avelar et al., 2023; Leal Filho et al. , 2023); low student engagement and awareness (Leal Filho et al., 2023); resource and infrastructure constraints (Do & Thinh, 2025); limited use of experiential and interdisciplinary learning (Monzó-Martínez et al., 2024); as well as the need for monitoring and evaluation tools (Edwards et al., 2020)

As we move forward, integrating models and modelling into policy and education will be essential to cultivating the systems literacy and collective agency needed to navigate the complexities of sustainable development. The *dim²ension* project aims to support mathematical modelling in the context of sustainability using a purposefully and effectively developed digital tool. Some of the issues listed above are consequently addressed through new approaches to teaching and learning mathematical modelling, which are made possible by the development and implementation of a highly motivating digital learning environment in which students can work on modelling tasks related to urgent SDG issues.

4. STE(A)M education

Chapter 4 has two aims. Firstly, it introduces and discusses STEM and STE(A)M education in general. Secondly, it highlights models and modelling in the context of STE(A)M.

4.1 What is STE(A)M?

STE(A)M education represents a comprehensive, interdisciplinary approach to teaching and learning that integrates the disciplines of Science, Technology, Engineering, the Arts, and Mathematics. It is grounded in the recognition that the challenges of the 21st century—ranging from climate change to technological innovation—require not only deep disciplinary knowledge but also the ability to synthesize and apply knowledge across domains (Le et al., 2023; Goos & Carreira, 2025; Ortiz-Revilla et al., 2020). STE(A)M is not simply a collection of subjects but a pedagogical paradigm that emphasizes the interconnectedness of knowledge and the development of critical competencies for responsible citizenship (Maass et al., 2019; Anderson & Makar, 2024).

STE(A)M education can also be thought about as a holistic framework for learning, where each component of STE(A)M contributes uniquely to a holistic educational experience. Science explores the natural world through inquiry and evidence-based reasoning. Technology focuses on the development and application of tools and systems to solve real-world problems. Engineering applies scientific and mathematical principles to design and build solutions, emphasizing iterative design and optimization. The Arts introduce creativity, imagination, and human expression, fostering innovation and empathy. Mathematics provides the analytical language and structure to model, quantify, and predict phenomena (Mass et al., 2019; Lillis, 2024).

This integration is not merely additive but synergistic. As emphasized in the Critical Questions in STEM Education volume, STE(A)M is best understood not as a collection of siloed disciplines but as a meta-discipline—a dynamic, interconnected system that reflects the complexity of real-world problems (Akerson & Buck, 2020). This perspective aligns with the view that STEM education should be "meaningful interdependence among all disciplines" (p. vi), rather than a superficial combination of subjects.

The book edited by Anderson and Makar (2024) further reinforces this view by highlighting how mathematics, often perceived as a subordinate tool in STEM contexts, can and should be foregrounded as a core discipline. Mathematics is not only foundational to the other disciplines but also offers unique ways of knowing, reasoning, and representing the world. When integrated meaningfully, mathematics enhances students' ability to engage with complex, interdisciplinary problems and supports the development of transferable skills such as modeling, argumentation, and abstraction.

When it comes to pedagogical approaches and educational goals, STE(A)M education is often implemented through inquiry-based learning, engineering design challenges, and project-based learning. These approaches encourage students to engage in authentic problem-solving, collaboration, and critical thinking. According to Baptista et al. (2023), such methods are effective in enhancing students' literacy across STEM fields and in fostering transferable skills.

Mathematical modelling plays a pivotal role in STE(A)M education by serving as a bridge between disciplines. It enables students to represent and analyze phenomena from science, technology, engineering, and the arts using mathematical tools (Goos et al., 2023; Kertil & Gurel, 2016; Stillman et al., 2023). Within the *dim²ension* project, for example, modelling tasks are designed to integrate multiple STE(A)M domains while addressing real-world issues such as those outlined in the Sustainable Development Goals (SDGs).

The book emphasizes that mathematical modeling is not only a method of application but also a means of deepening mathematical understanding. It allows students to engage in authentic practices such as reasoning, generalizing, and justifying, which are essential for both disciplinary learning and interdisciplinary problem-solving (Anderson & Makar, 2024). Moreover, when mathematics is purposefully integrated into STE(A)M tasks, it can shift from being a passive tool to an active site of learning, fostering both conceptual understanding and critical thinking.

The inclusion of the Arts in STEM—transforming it into STE(A)M—has been a subject of both enthusiasm and debate. As discussed in the book, the arts contribute more than aesthetic value; they are essential for cultivating creativity, empathy, and cultural awareness. These qualities are increasingly recognized as critical for innovation and for addressing complex societal challenges (Akerson & Buck, 2020).

In the context of STE(A)M, the arts also serve as a conduit for interdisciplinary thinking. They encourage students to explore problems from multiple perspectives, to communicate ideas through diverse media, and to engage in imaginative design processes. As highlighted in the LabSTEM project (Larsen et al., 2024), integrating the arts can also support responsible citizenship by connecting mathematical reasoning with ethical, cultural, and environmental considerations.

Despite its promise, STE(A)M education faces several challenges. These include disciplinary dominance (e.g., science or mathematics overshadowing other fields), lack of teacher preparation for interdisciplinary instruction, and institutional barriers to curriculum integration. The book highlights the need for teacher education programs to model integrated approaches and to support educators in navigating the epistemological differences between disciplines (Lederman & Lederman, 2020).

Moreover, the integration of STE(A)M must be intentional and pedagogically sound. As the editors caution, "true integration" requires more than thematic overlap; it demands coherence in learning goals, instructional strategies, and assessment practices. This includes ensuring that mathematics is not merely used for computation but is leveraged to support reasoning, modeling, and decision-making. Without such intentionality, there is a risk that mathematics becomes the "silent M" in STEM (Baldinger et al., 2020), undermining both disciplinary depth and interdisciplinary coherence.

4.2 Models and modelling in STE(A)M education

Models and modelling are increasingly recognized as foundational elements in integrated STE(A)M education. They serve not only as tools for representing and understanding phenomena within individual disciplines but also as bridges that connect the epistemologies, practices, and content knowledge across Science, Technology, Engineering, and Mathematics (Hallström & Schönborn, 2019; Hjalmarson et al., 2020). In this context, modelling becomes a unifying language that enables interdisciplinary learning and authentic engagement with real-world problems (Hallström & Schönborn, 2023).

From a mathematics education perspective, modelling is both a curricular goal and a pedagogical strategy. It involves the construction, analysis, and refinement of mathematical representations of extra-mathematical situations (Niss et al., 2007). This dual role—modelling as both content and vehicle—has been extensively discussed in the literature (Julie & Mudaly, 2007; Ärlebäck & Doerr, 2015). When applied to STE(A)M education, this perspective highlights how mathematical modelling can facilitate the integration of disciplinary knowledge and foster the development of transferable competencies such as abstraction, decomposition, and generalization (Shute et al., 2017).

The models and modelling perspective (Lesh & Doerr, 2003b) provides a robust theoretical framework for understanding how students develop and use models to make sense of complex systems. In this view, learning is conceptualized as model development, and modelling activities are designed to elicit, explore, and apply students' evolving conceptual systems. This perspective has been successfully applied beyond mathematics, including in science and engineering education (Diefes-Dux et al., 2006; Hjalmarson et al., 2008), and has been proposed as a viable strategy for achieving authentic and integrated STEM learning (Hamilton et al., 2008).

A particularly effective instantiation of this approach is the use of Fermi problems (FPs), which are open-ended estimation tasks that require students to decompose complex questions into manageable sub-problems and construct models based on assumptions and approximations (Ärlebäck, 2009; Albarracín & Gorgorió, 2014). FPs exemplify the modelling process in miniature and align closely with the six facets of computational thinking—decomposition, abstraction, algorithms, debugging, iteration, and generalization—identified by Shute et al. (2017) and adapted for STEM education by Hjalmarson et al. (2020).

Research shows that FPs can function as integrators of STEM disciplines by engaging students in modelling activities that draw on knowledge and practices from multiple domains (Ärlebäck & Albarracín, 2019). For example, in the "tin foil boat challenge," students estimate how many coins a foil boat can carry before sinking. This task involves physical principles (buoyancy, density), engineering design (boat construction), mathematical modelling (volume estimation, proportional reasoning), and technological tools (measurement and data analysis), thereby illustrating the interdisciplinary nature of modelling in STE(A)M contexts.

To support the implementation of FPs in educational settings, Ärlebäck and Albarracín (2019) introduced Fermi problem Activity Templates (FpATs), which visually represent the structure of a modelling solution, including the types of mathematical activities involved—guesstimation, measurement, data lookup, and statistical data collection. FpATs serve as both instructional and analytical tools, helping teachers design, scaffold, and assess modelling tasks that promote integrated STEM learning.

In summary, models and modelling are not only central to each of the STE(A)M disciplines but also serve as the connective tissue that enables interdisciplinary coherence. By adopting a modelling perspective—particularly through the use of FPs and FpATs—educators can design learning experiences that are authentic, integrative, and cognitively demanding. This approach not only makes the role of mathematics more visible in STEM education but also equips students with the competencies needed to navigate and address complex real-world challenges.

5. Task design

This chapter discusses important concepts and characteristics that underpin and permeate the design of the tasks and the learning environment surrounding the tasks in the *dim²ension* project. The chapter begins with an introduction on the role of tasks in mathematics education, focusing on mathematical modelling tasks to promote students' cognitive activation. Methods to cultivate critical thinking and creativity, fundamental human skills, in the context of the *dim²ension* project, are then discussed.

5.1 Tasks in mathematics education

The role and function of tasks in mathematics education is central to promoting student engagement and improving learning outcomes (Watson & Ohtani, 2015b). Tasks serve as the foundation for instructional strategies, guiding both instructional practices and student interactions (Scheja & Rott, 2024). Research has highlighted various frameworks and principles that inform effective task design, emphasising the importance of cognitive demands and social aspects in the learning environment (Watson & Ohtani, 2015a). One such framework is the Mathematical Tasks Framework (MTF), which categorises tasks based on their level of cognitive demand and provides insights into how tasks affect student learning outcomes, with higher cognitive demand tasks leading to deeper understanding and retention of mathematical concepts (Hsu & Yao, 2023).

Mathematical tasks are seen as mediating artefacts that facilitate student learning. According to Rezat and Strässer (2012), tasks serve as tools that mediate the interaction between students and mathematical content, fundamentally shaping the nature of the learning activity. This perspective is based on Vygotskian theory, which emphasises the role of tools in cognitive development. Often, the traditional didactic triangle (teacher-student-mathematics) has been reconceptualised to include mediating artefacts, forming a socio-didactic tetrahedron. This model highlights the interconnectedness of teacher, student, mathematics and tasks, and emphasises how tasks mediate the teaching and learning process (Clarke et al., 2014).

The work of Vaugelade et al. (2012) illustrates how the design of mathematical tasks has a significant impact on their effectiveness in mediating learning. Tasks that promote higher order thinking and problem solving are more likely to engage students deeply and improve their understanding of mathematical concepts. Research by Doyle (1998) and Hiebert and Wearne (1997) supports the idea that tasks designed to encourage higher order thinking lead to more meaningful learning experiences. In this vein, Radmehr (2023) proposes a framework for task design that includes principles such as inclusion, cognitive demand and the affective dimensions of learning. This framework serves as a reflective tool for educators to improve task quality and promote rich mathematical discussions.

Teachers are also found to play a crucial role in the delivery of mathematical tasks. Their decisions about task selection, implementation and adaptation influence how students engage with and learn from tasks. Studies have shown that teachers' discussions and preparations around tasks can provide important insights into how tasks mediate learning and how teachers can optimise their use (Vaugelade et al., 2012). The way students interpret and engage with tasks also mediates their learning. Tasks that are open-ended and allow for multiple solution paths can foster students' agency and creativity. One task-driven teaching method that positions tasks as central to the learning process and encourages active student participation and critical thinking.This approach is consistent with the Ordinary High School Mathematics Curriculum Standards, which advocate for teaching that stimulates student inquiry and communication, is discussed by Fu (2024). This approach is also consistent with the findings of Clarke et al. (2009), who emphasise the importance of considering students' intentions, actions and interpretations in task design and implementation Clarke et al. (2014).

In summary, mathematical tasks are central to the teaching and learning process, acting as mediating tools that shape students' mathematical experiences. Effective task design, teacher mediation, and student engagement are crucial for enhancing mathematical learning. However, there is ongoing debate about the balance between task complexity and student readiness. Some researchers argue that overly complex tasks may hinder learning, highlighting the need for careful consideration in task selection and design (Kaur & Chin, 2022). Modelling tasks are almost by definition complex tasks, and an aspect that is important to consider in the *dim²ension* project.

5.1.2 Mathematical modelling tasks

Mathematical modelling tasks are designed to integrate real-world problems into mathematics education, enhancing students' understanding and application of mathematical concepts. These tasks often involve creating mathematical representations of real objects or phenomena, which can be challenging for students due to the process of mathematization – the transition from reality to mathematical abstraction (Jablonski, 2024; Krutikhina et al., 2018). More generally however, modelling tasks typically require students to engage in one or more of the following activities: (1) understand a real-world situation; (2) formulate a mathematical model (mathematization); (3) solve the mathematical problem the model give raise to; (4) interpret and validate the results; and (5) communicate findings (cf. Niss & Blum, 2020).

Although modelling tasks are a subset of mathematical tasks in general, it could be argued that they particularly serve the following pedagogical goals:

- **Conceptual understanding**: Deepen understanding of mathematical concepts through application – using modelling as a vehicle for learning mathematical content (Julie & Mudaly, 2007),

- **Problem-solving skills**: Develop strategies for tackling complex, open-ended problems, essential for college and career readiness (Bock et al., 2015; Luczak & Erwin, 2023; Krutikhina et al., 2018),
- **Interdisciplinary learning**: Connect mathematics with science, economics, engineering, etc. (Bock et al., 2015; Serpe, 2025; Siller et al., 2023b),
- **Critical thinking**: Encourage evaluation of assumptions, limitations, and implications (Bock et al., 2015; Luczak & Erwin, 2023),
- **Engagement and motivation**: Increase relevance and interest by linking math to real life (Geisler & Rach, 2023;.Krutikhina et al., 2018).

Similar goals and demands are also discussed in Guerrero-Ortiz and Camacho-Machín (2022) in the context of teaching mathematics in a dynamic geometry system using modelling tasks and a modelling pedagogy.

In addition, as modelling tasks are connected to the real-world, there are particular challenges and difficulties in implementing modelling tasks. One such challenge is that students often struggle to transfer real-world problems into mathematical models, particularly when it comes to estimating, measuring and interpreting data (Bukova-Güzel, 2011; Jablonski, 2024). Another challenge relates to the preparation required of teachers. Teachers must be knowledgeable about the modelling process to ensure the effective implementation of modelling tasks (Brown et al., 2024; Durandt, 2021) and possess the skills needed to assess students' modelling competencies, which involves evaluating their ability to make assumptions, compute solutions and reason mathematically (Aydogan Yenmez, 2017; Leong & Tan, 2020).

The complex relationship between modelling tasks and the real world manifests itself differently for teachers and students alike. This is despite the fact that the modelling tasks presented in schools are often considerably simplified compared to actual modelling tasks. One characteristic manifestation is the complexity and contextualisation of the tasks, namely that real-world problems often involve more complex and dynamic variables than those presented in regular educational settings (Sokolowski & Banks, 2012; Akerele, 2017). A second manifestation has to do with practical constraints due to in real-world scenarios, modellers must deal with the fact that data, time, and resources are limited (Mazumdar et al. 2024, Wang et al., 2024), which are often simplified in educational tasks Vos (2011, 2015). A third manifestation stems from the interdisciplinary nature of real-world problems which hence require interdisciplinary approaches and collaboration, which can be challenging to replicate in a classroom setting (Palmgren & Rasa, 2024; Serpe, 2025).

Several taxonomies have been proposed for categorising modelling tasks based on their complexity and the skills they develop. A common starting point for formulating such a taxonomy is a modelling cycle (cf. Blum & Leiß, 2007), with the framework outlining the steps involved in the modelling process (e.g.

understanding the problem, simplifying it, solving the model and interpreting the results) aligned with this cycle. Durandt (2021) is an example of this approach. An alternative approach is to ground a taxonomy of modelling tasks in the notion of competency, focusing on the specific competencies students need to develop, such as making assumptions, computing solutions and reasoning mathematically (Brown et al., 2024; Leong & Tan, 2020). A third approach is to prioritise task authenticity (cf. Vos, 2011; 2015), as discussed by Kaiser and Sriraman (2006). They propose a classification based on the degree of authenticity and complexity, separating tasks into three categories: *intra-mathematical* (purely mathematical with no real-world context); *pseudo-real modelling* (real-world context is superficial or contrived); and *authentic modelling* (reflects real-world complexity and requires genuine modelling). A fourth approach is to focus on cognitive demands and student engagement, distinguishing between *routine modelling*, which is characterised by familiar contexts and procedures, and *non-routine modelling*, which involves more novel, complex and open-ended problems (Galbraith & Stillman, 2006).

Maaß (2010) developed a systematic and multi-dimensional classification scheme for modelling tasks, aiming to support both the design and selection of tasks tailored to specific educational goals and student groups. This scheme synthesizes prior classifications and introduces a structured framework grounded in modelling theory and design research. Maaß (2010) identifies nine classification dimensions, each with subcategories. The classification dimensions are displayed in Table 5.1.

Table 5.1. Multi-dimensional classification scheme for modelling tasks (Maaß, 2010)

Dimension	Examples of Categories
1. Modelling Activities Required	Whole process, interpreting, validating, mathematizing
2. Nature of Data	Superfluous, missing, inconsistent, matching
3. Relationship to Reality	Authentic, close to reality, embedded, artificial, fantasy
4. Contextual Situation	Personal, occupational, public, scientific
5. Type of Model	Descriptive, normative
6. Representation Format	Text, picture, material, real situation
7. Openness of Task	Closed (ascertaining), open problem, reversal, complex
8. Cognitive Demand	Extra-mathematical modelling, reasoning, representation use
9. Mathematical Content	Area (e.g., geometry), school level (e.g., primary, secondary)

Maaß's (2010) taxonomy complements and extends existing frameworks by offering a multi-faceted lens for task analysis and emphasizing target group adaptation. It in addition highlights the importance of task features (e.g., data type, openness) in shaping learning outcomes and encourages reflective task design that considers both cognitive and affective dimensions of learning. All four of these aspects are characterises of modelling tasks that are in cooperation in the task design farmwork adapted in *dim²ension* project.

In summary, mathematical modelling tasks are a powerful educational tool, fostering essential skills and interdisciplinary learning. However, careful consideration of the differences between educational and real-world contexts is required for their implementation. Effective teacher preparation and robust assessment frameworks are crucial for maximising the benefits of modelling tasks in education (Brown et al., 2024; Durandt, 2021; Siller et al., 2023b). Nevertheless, most taxonomies and design frameworks for modelling tasks in schools could be improved to make them more meaningful and realistic. For example:

- **Use authentic data and contexts** such as climate data or social issues.
- **Encourage open-ended exploration** and multiple solution paths.
- **Integrate digital tools** to simulate real-world modelling environments.
- **Foster interdisciplinarity** to reflect real-world problem-solving.

The *dim²ension* project aims to address this issue by providing a resource and tool to facilitate the implementation of modelling tasks in schools.

5.2 Critical thinking

Critical, or reflective thinking, is defined as a deliberate, self-regulated process of judgment that facilitates interpretation, analysis, evaluation, and inference, guided by evidentiary, conceptual, and methodological considerations (Dewey, 1933; Glaser, 1941; Lipman, 1991; Ennis, 1987). Cultivating critical thinking is among the most essential goals of education and one of its most valuable outcomes (Van Gelder et al., 2001; 2004). Consequently, fostering critical thinking should be a primary objective of all educational activities at every level (Nickerson, 1987; Kitchener & Fischer, 1990).

The fundamental skills associated with critical thinking include identifying problems, devising methods to solve them, gathering and organizing information, using language accurately, interpreting and evaluating data, drawing and reevaluating conclusions, and adapting opinions based on new experiences (Glaser, 1941; Beyer, 1995).

The essence of critical thinking begins with the act of thinking for a specific case, which presupposes a state of knowledge, confusion, or doubt (Dewey, 1933). Critical thinking emphasizes the individual's ability to decide what to believe or how to act when confronted with a challenge or a problematic situation requiring resolution (Ennis, 1989). It promotes self-improvement through the continuous evaluation and refinement of thought processes (Paul, 1990).

A fundamental characteristic of critical thinking is its openness to revision, marked by its capacity for self-correction (Lipman, 1944). Critical thinkers exhibit a willingness to revise their views, particularly when reflection demands reexamination and adjustment. Critical thinking also necessitates self-regulation of the thought process while requiring the formulation of coherent and thorough arguments, a task indicative of advanced mental functioning (Resnick, 1987). In summary, individuals who think critically present clear arguments, exercise prudence in judgment, approach complex issues methodically, and demonstrate cognitive flexibility.

Critical thinking abilities and skills, as complex cognitive processes, are approached by goals in the highest categories of the Bloom's taxonomy. The following examples of critical thinking educational objectives are from the condensed version of Bloom's taxonomy (Bloom et al. 1956: 201–207) retrieved from Hitchcock's article for critical thinking, in the "The Stanford Encyclopedia of Philosophy" (Hitchcock, 2024):

- *analysis objectives*: ability to recognize unstated assumptions, ability to check the consistency of hypotheses with given information and assumptions, ability to recognize the general techniques used in advertising, propaganda and other persuasive materials
- *synthesis objectives*: organizing ideas and statements in writing, ability to propose ways of testing a hypothesis, ability to formulate and modify hypotheses
- *evaluation objectives*: ability to indicate logical fallacies, comparison of major theories about particular cultures

Critical thinking and problem solving are deeply interconnected, forming the cornerstone of the educational methodology known as "Problem-Based Learning" (Facione et al., 2000; Choi et al., 2014; Carter et al., 2017). The following correlations illustrate their relationship (Paul et al., 1997):

- Problem solving inherently depends on critical thinking.
- Advanced critical thinking skills greatly enhance problem-solving effectiveness.
- Problem solving serves as a practical application of critical thinking, while critical thinking acts as an indispensable tool for achieving successful problem-solving outcomes.

When these interdependent processes are integrated, they produce significantly better results.

Problem-Based Learning is defined as an educational methodology where learners begin by engaging with a posed problem, striving to understand or resolve it (Barrows & Tamblyn, 1980; Barrows, 1986). This approach enhances learning by challenging learners to address ill-structured problems that simulate real-world scenarios (Hung et al., 2008). The problems presented to learners are designed to reflect authentic situations, bridging theoretical knowledge with practical application (Paget, 2004; Ahlfeldt et al., 2005).

Problem-Based Learning can serve as a logically coherent framework for the majority of mathematical modelling educational tasks. In the context of critical thinking and mathematics, mathematics serves as an effective tool to train students in critical thinking, as solving many mathematical problems inherently requires critical analysis. However, traditional mathematics instruction often focuses solely on manipulating mathematical concepts, failing to encourage students to engage in critical thinking (Schafersman, 1991).

To foster critical thinking when designing problem-based learning activities, the following principles should be considered:

- Problems should mirror real-world situations (Honebein et al., 1993).
- Learners are best engaged with poorly structured, real-world problems, as these reflect the complexity of actual challenges (Ahlfeldt et al., 2005; Paget, 2004).
- Learners' prior knowledge is critical to the learning process. By connecting new material to previous knowledge and experiences, learners create their own understanding, which supports the consolidation of new concepts (Savery & Duffy, 1995).
- The learning environment should encourage reflection. Addressing problems that activate critical and creative thinking helps learners deepen their understanding of their own thought processes. The teacher-facilitator plays a key role in fostering reflection, offering guidance, and monitoring learners’ comprehension of concepts and issues (Engle, 1997).

Engaging students in mathematical modelling problems is an effective approach to fostering and enhancing critical thinking. This process requires students to decompose/analyse problems into manageable components (Blum and Borromeo Ferri, 2009), formulate hypotheses, explore solutions (Niss et al., 2019), justify their reasoning (Frejd & Vos, 2023), and incorporate feedback to refine their approaches, thereby enriching their learning experiences (Buchholtz & Mesrogli, 2013). In essence, the practice of mathematical modelling in the classroom provides a dynamic platform for students to develop critical thinking skills. Moreover, strengthening critical thinking can also be achieved through activities that integrate Cognitive, Socio-Emotional, and Behavioral Learning (Gonzalez, 2023).

5.3 Creativity

Creativity is defined as the process of producing work of any kind that is both valuable and original (Csikszentmihalyi, 1999; Runco, 1997, 2000; Sternberg & Lubart, 1996; Gardner, 1993). The primary goal of creative thinking is to empower individuals to think beyond conventional boundaries, partially stepping away from common reasoning, standard ideas, and established procedures. It encourages imagination, serendipity, and the discovery of multiple and alternative solutions in given situations (Candy, 1997; Schlange & Juttner, 1997).

From the earliest studies of creative thinking, divergent thinking has been identified as a critical component of creativity, distinct from convergent thinking (Guilford, 1956). Creativity, and by extension divergent thinking, is developed and assessed based on the following abilities to (Torrance, 1966; 1990):

- generate many ideas (fluency of ideas).
- produce diverse ideas (flexibility of ideas).
- create unique or unusual ideas (originality of ideas).
- generate detailed ideas (elaboration of ideas).

These aspects of creativity can be taught, practiced, and enhanced through proven techniques that stimulate creative abilities, ideas, and outcomes (Mansfield et al., 1978; Parnes & Brunelle, 1967; Rose & Lin, 1984; Taylor, 1972). Moreover, creative thinking can enrich various learning contexts, fostering students' acquisition of knowledge and skills in dynamic and engaging ways (Starko, 2005; Cropley, 2001).

For the successful development of creative thinking and its application as a springboard for various educational activities, it is essential to establish conditions that foster: a pleasant and engaging environment, open communication, trust, tolerance for failure, utilization of external information sources, independence, initiative-taking, participatory decision-making, and experimentation with new ideas (Sefertzi, 2013; VanGundy, 2005).

The ability to visualize cognitive processes enhances students' creative thinking across all educational activities. "Insight" refers to the internal understanding of a problem's solution or the strategy leading to it, which emerges when individuals explore solutions to loosely structured and ambiguous problems (Sternberg & Lubart, 1991). By employing intuitive thinking, problem data is transformed into forms more comprehensible to students, facilitating their search for solutions to educational challenges (Guilford, 1967a). Consequently, teachers should incorporate vaguely defined, open-ended problems into learning activities to promote intuitive thinking and, by extension, student creativity (Jacobs & Dominowski, 1981; Martinsen, 1995).

Equally important is teaching students to explore their cognitive resources, empowering them to gain greater control over their thinking (Guilford, 1975). Teachers play a vital role in guiding students to understand creativity metacognitively during the educational process (Davis, 1991). Adopting a metacognitive approach to creativity fosters awareness of creative behaviors, demystifies creative outcomes, and enhances the generation of innovative ideas and products (Davis, 1991).

While many educators equate creativity with problem-solving, others view certain aspects of creativity as independent of problem-solving contexts (Runco

& Chand, 1995). This divergence stems from differences in how problems are presented to students and the observation that some creative outputs reflect personal expression rather than problem resolution (Csikszentmihalyi, 1988; Dudek & Cote, 1994; Runco, 1997). Creative thinking involves not just solving problems but also recognizing them, problem posing, designing solutions, and refining those solutions (Runco & Chand, 1995).

Problem identification is a critical stage in the creative process, serving as the foundation for subsequent solutions. Moreover, the quality of the identified problem significantly influences the quality of its solutions (Runco & Okuda, 1988). A creative solution is, therefore, the result of a well-conceived creative problem (Getzels, 1975). The ability to identify problems distinguishes creativity from problem-solving. Csikszentmihalyi and Getzels (1971) highlighted the distinction between "given problem situations" and "problem search situations," noting differences in the amount of initial information, the clarity of solution methods, and consensus on correct solutions.

On the other hand, although there is a wide variety of problem situations that can be presented to students, not all are suitable for fostering creative thinking. Goffin and Tull (1985) proposed a set of guiding questions to assist teachers in selecting appropriate problems for students to solve. Some of these questions include:

- Is the problem interesting?
- Can the problem be solved in multiple ways?
- Does the effort to solve the problem encourage the generation of new ideas?
- Can the actions leading to the solution be evaluated upon completion?

To conclude, creativity can be nurtured through exploration, discovery, and creative problem-solving, an indirect yet effective teaching method (Feldhusen & Treffinger, 1980). Educators emphasize the need for students to develop a broad range of problem-solving skills (Anderson & White, 2004). Structured opportunities in the classroom can foster these skills, but even isolated instances of problem identification and resolution stimulate creativity (Davis & Rimm, 1985; Subotnik, 1988). By integrating fluency, flexibility, elaboration, and originality into creative problem-solving, students' creative potential is further enhanced (Feldhusen & Treffinger, 1980). Finally, using realistic problems instead of hypothetical ones has been found to enhance creativity to a greater extent (Sternberg & Lubart, 1991).

Mathematical modelling tasks serve as a powerful tool for fostering creative thinking by encouraging the generation of innovative ideas (Lu & Kaiser, 2022), supporting the exploration of multiple solutions to problems (Vorhölter et al., 2014), and promoting knowledge discovery through experimentation (Bonotto & Santo, 2015). These activities specifically nurture divergent thinking, a critical element in the development of creativity (Durandt et al., 2022), while also engaging students in imaginative and flexible reasoning. This approach allows for unconventional problem-solving strategies, often leading to new ideas or discoveries (Cai et al., 2025).

5.4 Cognitive activation

Cognitive student activation refers to the learner's mental engagement facilitating a deeper exploration of the subject matter within an educational context. As a core component of teaching quality, cognitive activation is regarded as a crucial factor in initiating student learning processes. The empirical effects of cognitive activation are well documented in the relevant literature (Groß-Mlynek et al., 2022). Klieme et al. (2001), identified three fundamental dimensions to describe the quality of teaching mathematics. One of these is the degree of cognitive challenge given to students through tasks and classroom discourse.

From a constructivist perspective, effective instruction is characterized by its capacity to foster deep understanding through cognitive activation (Kunter & Voos, 2013). Within the COACTIV study's theoretical framework, cognitively activating instruction is defined as that which stimulates goal-oriented cognitive engagement with lesson content (Baumert et al., 2013). Rather than merely promoting behavioural activity, such as freedom of movement or hands-on manipulation of materials, cognitively rich environments aim to modify, extend, and reorganize students' existing knowledge structures. This dimension of instructional quality—often referred to as "content-rich instructional practice"—involves designing learning tasks that build on prior knowledge, challenge conceptual understanding, and encourage learners to apply their knowledge in novel contexts. For instance, complex tasks that require students to reinterpret known concepts or transfer their understanding to unfamiliar problems are considered cognitively activating. Furthermore, the way tasks are implemented plays a crucial role (Kunter & Voos, 2013); cognitively rich implementation may include structured classroom discussions in which students evaluate the soundness of their solutions or engage with diverse problem-solving strategies, thereby cultivating cognitive independence.

Empirical research has demonstrated that the potential for cognitive activation is a strong predictor of student learning gains (Praetoris et al., 2018). However, classroom observations often reveal a mismatch between the cognitive demands of assigned tasks and the level at which they are actually implemented during instruction. Theoretical underpinnings of this construct draw from both cognitive and socioconstructivist traditions (Praetoris et al., 2018). Cognitive psychology emphasizes the role of cognitive conflict in promoting conceptual change, whereby discrepancies between existing knowledge and new information prompt the reorganization of mental frameworks. Meanwhile, socioconstructivist approaches highlight the importance of social interaction and discourse in the construction of knowledge.

Learning is seen as a collaborative process, where exposure to contrasting perspectives and engagement in argumentation deepen conceptual understanding. Thus, instruction that facilitates both individual cognitive engagement and dialogic interaction is essential for fostering meaningful learning in mathematics.

5.4.1 Cognitive processes in mathematical modelling

The expectation of this specific project is to activate one or more cognitive processes from the student's perspective. Some of the main high level cognitive processes that may triggered during mathematical modelling are the following:

- **Problem Decomposition and analysis**: When complex problems are broken down into additional subproblems that are easier to understand, it is easier to understand and come up with other solutions. This is best exemplified by decomposition with respect to systemic thought and disposition to solve the interconnections of problems (Stillman et al., 2019).
- **Abstraction**: In mathematical modelling, abstraction enables learners to simplify complex real-world scenarios by focusing on essential features while disregarding extraneous details (Blum & Leiß, 2007; Lesh & Doerr, 2003a; Gillmor et al., 2015). By engaging with abstraction, students develop the ability to identify key variables and relationships, fostering deeper cognitive skills and an enhanced understanding of mathematical concepts. This process not only supports the construction of more accurate and meaningful models but also equips learners with the capacity to generalize insights across contexts, a vital skill in problem-solving and reasoning.
- **Logical reasoning** is a core skill in mathematical modelling, allowing people to analyse problems, find connections, and draw sound conclusions. It involves using both deductive reasoning, which moves from general principles to specifics, and inductive reasoning, which builds generalizations from observations. This process helps create models that accurately represent real-world situations and ensures their reliability through careful validation and proof (Brenner, 2010).
- **Reasoning under uncertainty** is equally important, as many real-world problems involve missing or unclear information. This type of reasoning uses tools like probability and statistics to make educated guesses and informed decisions despite the unknowns. Incorporating uncertainty into mathematical models helps address variability and risk, leading to solutions that are practical and adaptable to real-world conditions (Ristic, 2019).
- **Critical thinking and problem-solving** are foundational skills in mathematical modelling, encouraging students to analyze complex problems, construct hypotheses, and evaluate solutions using evidence-based reasoning (Dewey, 1933; Glaser, 1941). These processes are deeply interconnected, as effective problem-solving relies on critical thinking to

interpret and refine problem structures (Paul et al., 1997; Niss & Højgaard, 2019). Problem-Based Learning (PBL) serves as a practical framework for fostering these skills, engaging students with real-world, ill-structured problems that stimulate analytical and reflective thinking (Barrows & Tamblyn, 1980; Hung et al., 2008). Through mathematical modelling tasks, learners decompose problems, justify their reasoning, and incorporate feedback, promoting both cognitive flexibility and deeper learning (Blum & Borromeo Ferri, 2009; Frejd & Vos, 2023). This approach connects abstract mathematics to practical applications, preparing students for dynamic, real-life challenges (Engle, 1997; Buchholtz & Mesrogli, 2013).

- **Creativity and Innovation**: Creativity plays a vital role in mathematical modelling, as it empowers students to approach complex problems with flexibility, originality, and divergent thinking (Guilford, 1956). By engaging with loosely structured and open-ended problems, learners are encouraged to explore multiple solutions, fostering both imagination and innovative reasoning (Runco, 1997; Torrance, 1990). Mathematical modelling tasks, which often involve real-world scenarios, provide an ideal platform for nurturing these skills. They enable students to experiment, pose problems, and refine their solutions, thereby enhancing their ability to generalize insights and apply creative thinking across diverse contexts (Bonotto & Santo, 2015; Lu & Kaiser, 2022).
- **Transfer of Knowledge**: Instead of viewing mathematics as an isolated industry, incorporating abstract mathematical disciplines with economics, physics and others brings a better understanding across disciplines as practical application validates theoretical knowledge (Ledezma et al., 2024).
- **Metacognition**: Reflective thinking allows students to evaluate their strategies and refine their results. It also fosters to some extent the analysis of the assumptions of a model which results in greater self-insight and learning flexibility. (Verschaffel et al., 2020b).

The nature of mathematical modelling tasks presents aspects that increase students' cognitive activation. Such aspects include:

- Real-World Context: Real, complex, and uncertain problems increase interest and encourage critical thinking (Borromeo Ferri, 2018).
- Combination of Mathematical Concepts: Algebra, geometry, and statistics combined lead to interdisciplinary and holistic understanding (Niss & Højgaard, 2019).
- Collaboration and Communication: Interactions between the peers and building up solutions contribute to understanding through discussion and critique (Teig et al., 2019).
- Iterative Feedback: Iterative learning and critical reflection are provided for through on-going cycles of enhancement and revision (Verschaffel et al., 2020b).

From the above, there is a strong connection between cognitive activation and the design of mathematical modelling tasks. Teachers can create active and cognitively stimulating learning environments by integrating real-world contexts, critical thinking, creativity, problem solving, sustainability, interdisciplinarity, data handling and interpretation, scalability, estimation, reflection, collaboration, complexity, authentic data, mathematical concepts, openness, differentiation, process management and interactive elements. These task designs do not only lead to improved mathematics skills but also transferable attributes that are important for creativity and problem-solving in real-world contexts.

5.4.2 Cognitive activation by *dim²ension* tasks

In the following section, we describe the potential for cognitive activation in tasks designed within the *dim²ension* project. First, we focus on modelling tasks in general. Secondly, specific types of modelling tasks that have recently been discussed in the literature and that are relevant to the *dim²ension* project are discussed below in terms of their potential for cognitive activation.

Generally spoken, modelling tasks require students to apply mathematics to solve issues directly relevant to real-life circumstances. This need of transferring real-world situations to mathematics and vice versa significantly contributes to cognitive activation:

Modelling tasks are designed to provide all the necessary information and data, enabling students to focus on applying their knowledge effectively. A critical aspect of these tasks is decomposition, which involves breaking down complex problems into manageable subtasks. This approach enhances students' understanding of intricate systems and promotes systemic thinking (Stillman et al., 2010). Additionally, abstraction plays a key role, allowing students to simplify real-world scenarios by isolating essential features and disregarding irrelevant details. Through this dual process, students learn to relate abstract mathematical ideas to realistic applications, critically evaluate provided data, and devise efficient problem-solving strategies. These activities bridge the gap between mathematical theory and real-world contexts, fostering a deeper understanding of mathematical concepts and developing transferable problem-solving skills (Blum & Leiß, 2007; Kaiser & Sriraman, 2006; Lesh & Doerr, 2003b).

These considerations about the potential for cognitive activation apply to modelling tasks in general. As in the recent literature, Fermi problems (e.g., Ärlebäck & Albarracín, 2022) and interdisciplinary modelling tasks (e.g., Maas et al., 2019) have been highlighted as specific modelling task types. As these task types are important in the *dim²ension* project (see section 7.1), their potential for cognitive activation is discussed below.

Due to the emphasis on estimation and order-of-magnitude calculations, Fermi problems also evoke other cognitive processes in the learner. With just enough data, even minimalist data, these types of challenges indeed test a student's ability to make educated guesses and reasonable approximations. Due to their nature, open-ended and requiring one to come up with a plausible solution by reason, Fermi problems develop creativity. By asking students to defend their assumptions and assess their estimations, these assignments also promote critical thinking. Students who participate in these cognitive exercises improve their logical reasoning abilities and develop self-assurance while dealing with uncertainty in problem-solving situations (Lesh & Doerr, 2003a; Verschaffel et al., 2000a).

With the integration of mathematics with science, technology, engineering, and, in some cases, the arts, *interdisciplinary STE(A)M tasks* give an additional boost to cognitive activity. When executing these modelling tasks, students have to relate mathematical concepts to other subjects, and the tasks put a strong emphasis on interdisciplinary thinking. Students learn how mathematics can describe and analyze complex systems in detail by working with phenomena from a variety of academic disciplines. Also, since students need to simplify complex phenomena into tractable mathematical models, these tasks often require abstraction. Most of the interdisciplinary tasks are group-based in nature, and this encourages peer interaction and articulation of concepts, which strengthen communication and collaboration skills as well as deepen understanding. These tasks also motivate students and raise their feeling of accomplishment by finding solutions to real-world problems (Gravemeijer & Doorman, 1999; Stillman et al., 2019).

To summarize, mathematical modelling tasks can compel students to employ critical and creative applications of mathematical knowledge, establishing connections between multidisciplinary concepts through the resolution of real-world problems. Particularly, the modelling task types used in the *dim²ension* project, i.e., basic modelling tasks, Fermi problems and STE(A)M modeling tasks (see section 7.1) have the potential to facilitate high order cognitive activation where necessary knowledge and skills useful in real situations are acquired next to the growth of mathematical competence.

6 Mobile and technology-enhanced learning

In recent years, mobile learning and digital platforms have transformed how students engage with mathematics, offering new forms of interaction, collaboration, and context-rich experiences. Recent research has demonstrated the significant potential of digital technologies in supporting mathematical modelling in education (Cevikbas et al., 2023; Siller et al., 2023a). These tools not only foster students' engagement with real-world problems but also enable flexible, individualized learning environments. Among them, mobile technologies have received particular attention due to their portability and ability to support outdoor and context-sensitive modelling activities (Buchholtz, 2021; Fesakis et al., 2018; Jablonski et al., 2023; Ludwig & Jesberg, 2015).

Whether used in outdoor settings or classroom environments (Barlovits et al., 2024; Milicic et al., 2021), mobile learning (m-learning) represents a powerful approach within the broader field of Technology-Enhanced Learning (TEL). It enables learners to access knowledge anytime and anywhere using smartphones, tablets, and other handheld devices (Handal et al., 2016). In the context of mathematical modelling, this flexibility allows learners to engage with real-world data in situ, record observations, and construct models directly in their environment.

Furthermore, integrating web portals and educational applications can amplify the flexibility, interactivity, and personalization of learning experiences (Dabbagh & Bannan-Ritland, 2005). Additionally, previous projects such as MathCityMap (Ludwig & Jesberg, 2015), <colette/> (Läufer et al., 2024), and ASYMPTOTE (Barlovits et al., 2022a) have significantly contributed to the understanding and application of the development of multi-tiered systems that combine web portals and m-learning technologies in complex but effective learning environments, particularly in mathematics education.

This section explores the theoretical foundations and practical applications of m-learning, focusing on the role and function of web portals and apps in supporting education. Special attention is given to the teaching and learning of mathematics, highlighting the benefits, challenges, and potential of these technologies in this domain.

6.1 General principles of mobile learning

According to Winters (2007), perspectives on m-learning can be categorized into four broad areas:

1. Technocentric: Focuses on the use of mobile devices as learning tools.
2. Relationship to Online Learning: Highlights m-learning as a flexible extension of online learning.

3. Enhancing Traditional Education: Views m-learning as an enriching tool for formal education settings.
4. Learner-Centered: Concentrates on the mobility of the learner, emphasizing the flexibility to learn in various contexts.

These perspectives underline the versatility of m-learning in various TEL settings, from face-to-face interactions to fully online courses. Over time, there has been a noticeable shift from techno-centric to learner-centered understandings of m-learning, as also emphasized by Yamamoto (2013). She notes that m-learning allows for educational processes “without breaking apart from life,” highlighting its potential to integrate seamlessly into learners' everyday activities and routines.

M-learning can be broadly defined as "any educational provision where the sole or dominant technologies are handheld or palmtop devices" (Traxler, 2005, p. 262). It shifts learning environments from static, desktop-based setups to dynamic, context-aware experiences. In this vein, Crompton (2013, p. 4) defines m-learning as "learning across multiple contexts, through social and content interactions, using personal electronic devices," reinforcing the personalized and interactive nature of m-learning.

Yamamoto (2013) further emphasizes that m-learning removes traditional barriers to education, such as limitations of location, time, and resource availability. More importantly, it provides learners with interactive, personalized, and context-sensitive learning opportunities, fostering engagement and accessibility. This context sensitivity, related to parameters such as location, time, or learner state, is an essential feature of m-learning. Recent reviews such as Suarez et al. (2018) also underline the wide-ranging pedagogical potentials of m-learning, particularly its ability to support inquiry-based learning experiences.

6.2 Mobile learning in mathematics education

Recent research supports the integration of m-learning in K–12 education. Crompton and Burke (2020) conducted a systematic review of m-learning in PreK–12 settings using the SAMR (Substitution, Augmentation, Modification, Redefinition) framework, identifying pedagogical benefits such as increased student engagement and opportunities for more transformative, student-centered instruction. While their study covered multiple subjects, their findings underscore the broader potential of mobile technologies to enhance learning experiences.

This potential is particularly significant in mathematics education, where mobile tools offer dynamic and visual affordances that can support students' understanding of abstract concepts and promote more interactive forms of learning.

The use of mobile technologies in mathematics education has gained increasing attention over the past decade. Calder et al. (2018) argue that the interactive and visual affordances of mobile devices transform mathematical activities, enabling learners to engage with complex concepts in innovative ways. For example, graphing calculators and dynamic geometry apps provide learners with immediate, visual feedback, enhancing their understanding of abstract concepts.

Moreover, Burke et al. (2021) provide a valuable theoretical lens through which m-learning in mathematics can be understood. Drawing on Activity Theory and the Technological Pedagogical Content Knowledge (TPACK) framework, they emphasize that m-learning is most effective when it facilitates collaborative, inquiry-based learning. Well-designed digital tasks should promote critical engagement, argumentation, and reflection, rather than simply serving as content delivery tools. Their study also highlights important disciplinary differences: while science teachers often integrate mobile technologies in tool-oriented ways, mathematics teachers are more likely to adopt structured, didactic, or project-based approaches, often limiting opportunities for open-ended exploration. This underscores the importance of developing discipline-specific pedagogical frameworks and supporting teacher readiness for mobile integration in mathematics.

Despite these advancements, challenges remain. Designing effective educational scenarios that leverage mobile technology for meaningful mathematical engagement can be difficult for educators. While earlier studies such as Botzer and Yerushalmy (2007) and Daher (2010) demonstrated the potential of m-learning for authentic, real-world problem-solving and collaboration, rapid technological developments have rendered some of their contexts outdated. However, the pedagogical principles they explored—such as fostering active engagement, collaborative sense-making, and authentic task design—remain highly relevant.

The COVID-19 pandemic accelerated the shift to digital learning and exposed both the potential and limitations of m-learning in mathematics. Ferguson (2020) identified flexibility, interactivity, and personalization as key benefits, but also noted challenges such as cheating, isolation, and unequal access to technology, which continue to shape post-pandemic digital education strategies.

Projects like ASYMPTOTE (Barlovits et al., 2022a) respond to these pedagogical challenges by incorporating the Community of Inquiry (CoI) framework. By combining a mobile app for students with a task-management portal for teachers, ASYMPTOTE facilitates context-sensitive mathematical modelling and supports collaborative knowledge building in both classroom and outdoor environments.

In a broader perspective, Drigas and Pappas (2015) argue that m-learning should foster not only subject-specific understanding but also metacognitive and higher-order thinking skills. Their work supports a shift from content transmission toward constructivist and student-centered approaches in mathematics education.

Although m-learning technologies are increasingly adopted, comprehensive theoretical frameworks remain scarce, particularly in subject-specific contexts like mathematics (Calder et al., 2018). By integrating experiences and insights from m-learning applications, the project aims to shape a robust pedagogical framework that supports both teachers and learners. Building on these general pedagogical insights, the next section examines how m-learning specifically supports mathematical modelling.

6.3 Mobile learning in mathematical modelling

M-learning offers a wide range of advantages that are particularly relevant for mathematical modelling, as it promotes interactive, accessible, and contextually enriched learning environments. As Larkin and Calder (2016) argue, mobile technologies are reshaping when, where, and how mathematical learning occurs, enabling more exploratory and student-centred approaches. Numerous studies have emphasized the role of mobile technologies in enabling flexible, situated modelling activities, both in classroom and outdoor settings, thus expanding the boundaries of traditional instruction (Buchholtz, 2021; Fesakis et al., 2018; Jablonski et al., 2023; Ludwig & Jesberg, 2015; Barlovits et al., 2024; Milicic et al., 2021). These advantages include accessibility and flexibility, global connectivity, context-sensitive data collection, real-time feedback, interactivity, visualization, and augmented reality (AR) applications.

Among the most prominent benefits of m-learning in modelling is its accessibility and flexibility. Students can engage in modelling tasks anytime and anywhere, supporting self-directed and individualized learning experiences (Fabian et al., 2018). Global connectivity further enhances learning by allowing students to collaborate with peers, access empirical data, and use simulation tools and modelling platforms, such as MathCityMap, which exemplifies outdoor mathematical modelling using mobile devices (Siller et al., 2023a).

Mobile devices also enable context-sensitive learning through embedded sensors (GPS, accelerometers, cameras, etc.), allowing students to collect and analyze real-world data in situ (Chen, 2019). This situational engagement helps learners see the relevance of mathematics in solving authentic problems, such as environmental monitoring or financial forecasting. As shown by Cevikbas et al. (2023), integrating mobile technology into modelling activities empowers students to apply theoretical knowledge in concrete, meaningful contexts.

Interactivity and visualization are equally central to m-learning in modelling. Mobile apps provide dynamic tools for graphing, manipulating variables, and simulating complex systems, enhancing conceptual understanding. Augmented reality (AR) adds another layer by enabling exploration of three-dimensional models, grounding abstract mathematical ideas in concrete visual experiences (Ibáñez & Delgado-Kloos, 2018). Importantly, m-learning also supports real-time feedback, allowing students to iteratively refine their models based on immediate input (Grant, 2019). These affordances collectively foster deeper comprehension and support the development of key modelling competencies such as prediction, representation, and system analysis.

The benefits of m-learning are not only theoretical but have been demonstrated in practice. For instance, the ASYMPTOTE system (Barlovits et al., 2022b) leverages mobile and web-based technologies to support mathematical modelling across contexts. Fesakis et al. (2023) showed that its structured use of learning graphs, scaffolded hints, chat functions, and support tasks helps students overcome misconceptions, particularly in topics such as fractions. The platform's digital classroom features enable real-time interaction between students, teachers, and learning guides (LGs), promoting a responsive and collaborative modelling process.

In summary, m-learning significantly enriches the instruction and comprehension of mathematical modelling by facilitating dynamic, situated, and personalized learning experiences. It supports key competencies such as problem-solving, teamwork, creativity, and critical thinking, all essential to meaningful engagement with modelling tasks. However, as noted by Siller et al. (2023a) and Cevikbas et al. (2023), the pedagogical potential of m-learning in mathematical modelling is still underexplored. The *dim²ension* project addresses this gap by systematically investigating how mobile technologies can support effective, context-rich modelling experiences in both formal and informal learning environments.

6.4 The role of web portals and apps

Web portals and applications play a pivotal role in the delivery of m-learning. They serve as gateways to digital resources, enabling learners to access interactive content, participate in collaborative activities, and track their progress. According to Dabbagh & Bannan-Ritland (2005), the effective implementation of online learning models requires the identification of instructional strategies and learning methods suitable for web-based online learning.

Web portals offer centralized platforms for educational resources, integrating tools such as discussion forums, multimedia content, and assessment modules. They foster collaboration and provide a structured environment for learning. For instance, platforms like Moodle, ILIAS (n.d.) and edX (n.d.) have been widely adopted in educational settings to support blended and online learning.

Mobile software applications - Apps extend the functionality of web portals, offering tailored learning experiences. Apps like Duolingo (n.d.) for language learning and Photomath (n.d.), MathCityMap and Asymptote for Mathematics illustrate how mobile technologies can engage learners through gamification, real-time feedback, and intuitive interfaces.

Previous initiatives, such as MathCityMap, have demonstrated how mobile technologies can engage learners in interactive, location-based mathematics activities. MathCityMap allows students to explore mathematical problems in their physical environment, fostering connections between theoretical concepts and real-world applications (Ludwig & Jesberg, 2015). This project set the groundwork for the integration of context-aware technologies in mathematics education.

Similarly, the <colette/> project emphasized the potential of collaborative learning through web-based platforms. It provided a structured environment where learners could interact with digital tools, share ideas, and co-construct knowledge in mathematics (Milicic et al., 2021).

The ASYMPTOTE project expanded on these concepts, offering an innovative approach to adaptive and synchronous online education, leveraging m-learning to enhance interactivity and engagement. At its core lies the "digital classroom," a dynamic environment that enables real-time interactions among students and between students and teachers. This platform also provides tools for teachers to monitor their students' progress in real-time, fostering immediate feedback and personalized learning experiences. ASYMPTOTE bridges the gap between traditional and mobile education by emphasizing adaptive delivery, creating a seamless and interactive learning ecosystem (Barlovits et al., 2022b).

In addition to these projects, widely used tools such as Geogebra and Desmos exemplify the power of web-based applications in mathematics education. Geogebra offers dynamic geometry, algebra, and calculus tools (Sangwin, 2007), while Desmos provides an intuitive interface for graphing and exploring mathematical relationships (Liang, 2016). Collectively, these platforms illustrate the diverse and expanding landscape of mobile-supported mathematics education, one that *dim²ension* seeks to navigate and extend. The integration of these tools underscores the importance of well-designed tasks. The next section introduces the *dim²ension* project's pedagogical framework for task development in mobile-supported modelling.

7. *dim²ension* task framework

The *dim²ension* project aims to provide an openly accessible database of mathematical modelling tasks in the context of Education for Sustainable Development (ESD). To ensure high quality task design, the *dim²ension* project is grounded on research literature. Within the project, a task classification framework is developed. Furthermore, task design criteria for mathematical modelling tasks in the context of sustainability is derived.

7.1 *dim²ension* task types

Chapter 2 discussed and defined the general notion of modelling tasks. This definition is broad enough to encompass most, if not all, other subtypes of modelling tasks found in the literature. In order to fulfil the requirements and specifications of the *dim²ension* project, three types of modelling are distinguished: *Basic modelling tasks*, *Fermi problems*, and *STE(A)M modelling tasks*. These task types encapsulate an increase in the complexity and the level of originality of the work required of students, in the sense that

- in basic modelling tasks, given (simple) models are applied to simple real world problems/contexts/scenarios;
- in Fermi problems, the focus is more on the modelling- and open problem-solving process (alas emphasising simple mathematical ideas); and
- STE(A)M modelling tasks are full-fledged complex modelling tasks.

The following sections will provide more detail on these three task types and give examples.

7.1.1 Basic modelling tasks

Basic modelling tasks primarily involve using mathematics to solve a problem that is clearly and strongly linked to a real-life situation (cf. Stillman et al., 2019). Importantly, all the data and information needed to solve the tasks are provided within the tasks themselves, and no knowledge of STE(A)M disciplines other than mathematics is required to solve them. When tackling basic modelling tasks, students are required to use simple, pre-defined mathematical models to express a real-world situation in mathematical terms, as well as to interpret the resulting mathematics in the context of the real-world situation in which the task is set. When engaging with and solving basic modelling tasks, the main learning objectives for students are to move beyond using pure mathematics alone and to apply their mathematical knowledge to real-life phenomena (cf. Blum, 2015).

Example of a basic modelling task: ***Over-tourism on Rhodes Island***

Over-tourism on Rhodes Island

Travel has become more accessible and convenient than ever, leading to a significant rise in tourism in Greece, particularly on the island of Rhodes. In 2024, Rhodes experienced record-breaking tourist numbers, highlighting the growing popularity of the island as a travel destination.

Using the Visitor-to-Resident Ratio, investigate whether the number of tourists in 2024 exceeded the sustainable capacity of the island. Analyze whether the tourism levels that year placed significant pressure on the local community and infrastructure. If such pressures are identified, propose solutions to address these challenges and ensure sustainable tourism development.

Task image:	National Technical University of Athens (2017). *COMPDYN 2017 15-17 June 2017. The Island of Rhodes*. Link.
Task authors:	Maria-Anastasia Moustaka & Georgios Fesakis

The given task example can be considered a basic modelling task. Firstly, the visitor-to-resident ratio is calculated. This can be considered a simple, pre-defined model. The visitor-to-resident ratio is then computed based on real-world data found online.

The next step is to grasp, understand and interpret this ratio within a real-world context. What would the consequences for the local community and infrastructure be if the ratio were high? What would the consequences be if the number of visitors decreased? What might more sustainable tourism on Rhodes Island look like?

7.1.2 Fermi problems

Fermi problems are a specific type of modelling task that involves making reasonable estimates and educated guesses to perform order-of-magnitude calculations (Ärlebäck & Albarracín, 2022). One possible definition of a Fermi problem is "open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations" (Ärlebäck, 2009, pp. 331–332).

Advances in research on Fermi problems in recent years, made possible by increased access to data and information provided by the internet, have relaxed the chrematistics that quantities need to be estimated based on prior experience and personal knowledge alone (Ärlebäck & Albarracín, 2019). Rather, the numerical values required to solve a Fermi problem have been found to be achieved mainly through one of the following four activities: (1) *guesstimation*, which is a mental process involving rough approximations based on intuition, comparisons and prior experience; (2) *experimentation*, which involves conducting in- or out-of-school investigations or measurements to gather data; (3) *data search*, which involves looking up numerical information from external sources such as websites, books or databases; and (4) *statistical data collection*, which involves designing and conducting surveys or collecting data systematically in order to analyse it and derive estimates (Albarracín & Ärlebäck, 2019).

Following this slightly relaxed definition, students solving Fermi problems are required to divide the given task into manageable subtasks, as well as making reasonable estimates to obtain the necessary information, even in the absence of data. The primary learning objectives for students solving Fermi problems are to set up and work with a mathematical model and basic arithmetic, and to learn how to deal with situations where data is missing as well as how to conduct research to obtain the necessary information.

Example of Fermi problems: *Adele & Taylor Swift in Concert*

Adele & Taylor Swift in Concert

In August 2024, British pop star Adele performed ten concerts. The special feature: Instead of touring around the globe, Adele performed all of her concerts in Munich, Germany. The shows took place in an outdoor arena that had been built especially for these shows. In total, 730,000 people attended the concerts.

Based on an analysis of mobile data for the first show, we know that about 50% of the people travelled to Germany from abroad. Around 10,000 fans came from the UK and several thousand from Austria, the Netherlands and Poland. There were also 1,000 people from the USA as well as smaller groups of fans from Argentina, Australia, Brazil, Canada, China and India.

On the other hand, American pop star Tayler Swift toured the world in 2023 and 2024. In her so-called Eras tour, she played a total of 149 concerts all around the globe. In July 2024, she played a concert in Gelsenkirchen, Germany, in front of 60,000 fans. According to an analysis of mobile data, almost 12% of the fans came from other countries. American fans were the largest international fan group, accounting for more than 2% of all visitors. They were followed by fans with a relatively short journey, e.g. from the Netherlands or Belgium, and then by fans with a medium journey, e.g. from the UK or Austria. Smaller international groups of fans were from Australia, Canada, China and India.

Based on the given data, we can assume that a series of concerts in one place (such as Adele) dramatically increases the travelling of fans compared to a tour (such as Taylor Swift)

Estimate the amount of additional CO2 [in tonnes] for fan mobility that will be emitted for a single concert in an Adele concert series compared to a Taylor Swift world tour. For the purposes of this comparison, assume that both Adele and Taylor Swift performed in front of 73,000 fans.

Task image:	Lady Lotus (2023). *Adele performing at The Colosseum in Las Vegas on February 24*, CC BY-SA 4.0. Wikimedia Commons. Link. iHeartRadio (2023) *Taylor Swift at the 2024 Golden Globes*, CC BY 3.0. Wikimedia Commons. Link.
Task ressource:	Télefonica Germany (2024). *Mobilitätsanalyse belegt: Hälfte der Adele-Konzertbesucher in München reist aus dem Ausland an.* Link. Télefonica Germany (2024). *What mobile phone data reveals about the "Swifties".* Link.
Task authors:	Simon Barlovits

This task can be considered a Fermi problem since learners must create a model from scratch in order to address this real-world situation with incomplete information. To do so, they can extract information from the text, such as how many German fans attended Adele and Taylor Swift's concerts and which other countries had a large number of fans in attendance.

However, many estimations still need to be made. For example, how many kilometres did fans from Germany travel on average to the concerts? How many kilometres did fans from other European or non-European countries travel? What transport options did they choose depending on the distance travelled?

Based on these considerations, learners can use online calculators to estimate emissions related to the means of transport and travelling distance. The task therefore involves making an order-of-magnitude estimate based on realistic assumptions; only basic arithmetic is required.

Finally, learners must interpret the mathematical result in a real-world context. Why is there more international fan mobility for Adele's one-off concert than for Taylor Swift's world tour? How much more CO_2 is emitted by this fan mobility? What might the consequences be in terms of sustainability?

7.1.3 STE(A)M modelling tasks

STE(A)M modelling tasks are a specific type of modelling task that uses mathematics to describe phenomena in science, technology, engineering and arts (Goos et al., 2023; Stillman et al., 2023). When tackling STE(A)M modelling tasks, students are expected to apply mathematics more integrally to address scientific, social, political or economic problems, thereby connecting and combining knowledge from different disciplines. Consequently, the learning objectives for students engaged in STE(A)M modelling tasks are more advanced and demanding than those for basic modelling tasks and Fermi problems. Students engaged in STE(A)M modelling tasks are expected to use mathematics more broadly and flexibly to address scientific, social, political and economic problems, and to recognise modelling as a scientific method in STE(A)M disciplines. They should also further their knowledge in these disciplines and connect it to their mathematical knowledge.

Example of STE(A)M modelling task: *Carbon Footprint*

Carbon Footprint

The Carbon Footprint represents the area of forest land required to sequester anthropogenic carbon dioxide emissions. The calculation of the Ecological Footprint – and the Carbon Footprint in particular – uses mathematical models. Its calculation is detailed in a guidebook (see task resources), which also gives an example calculation for Hungary.

With the help of the guidebook, determine the Carbon Footprint of production of Hungary using a suitable mathematical model. Critically compare it against the result as indicated in the guidebook by finding reasons for possible differences.

Task image:	Pexels (2016). *Sand Fußabdrücke*, CC0 1.0. Link.
Task ressource:	Global Footprint Network (2025). *Glossary*. Link.
Task authors:	Hans-Stefan Siller & Alissa Fock, adapted from Siller et al. (2024c).

This task can be considered a STE(A)M modelling exercise, as it requires students to calculate the CO_2 emissions of countries using Hungary as an example. Here, learners explore which country sides have a stronger positive or negative impact on CO_2 absorption. To calculate a country's carbon footprint, its consumption and land use, among other factors, are taken into consideration.

Firstly, learners must understand and apply a given mathematical model of the ecological footprint of countries. Secondly, learners validate the model by researching, understanding and applying a second model. Thirdly, the two models are compared.

This task requires STE(A)M processes as follows: From an environmental perspective, learners must research and identify how different types of countryside contribute to a country's biocapacity. From an economic perspective, they must research and identify the impact of agricultural and industrial production. This may raise questions such as: What is the positive environmental impact of preserving forest or grassland areas? Why do industrialised states contribute much more to the global climate crisis than non-industrialised states? What might the negative political, economic, and social consequences be of preventing countries from preserving nature?

7.1.4 Connections between the task types

Table 7.1 summarises the description of *basic modelling tasks*, *Fermi problems* and *STE(A)M Modelling Tasks* given above and provides an example of each task type.

Table 7.1. Summary and examples of the three *dim²ension* modelling types of tasks

	Basic Modelling Tasks	**Fermi Problems**	**STE(A)M Modelling Task**
Task type	*Basic modelling tasks* aim at using mathematics to solve a problem strongly linked to a real-life situation (cf. Stillman, 2019). Importantly, all data/information needed to solve the tasks is provided in the task, and no knowledge of STE(A)M disciplines are needed per se to solve the task.	*Fermi problems* are a special type of modelling tasks. They aim at reasonable estimates and educated guesses to carry out order of magnitude calculations (Ärlebäck & Albarracín, 2022).	*STE(A)M modelling tasks* are a special type of modelling tasks. They aim to describe phenomena of science, technology and engineering (and arts) through mathematics (Goos et al., 2023; Stillman et al., 2023).
Process require-ments	Processing basic modelling tasks require students a) to use a more or less given and simple mathematical model to mathematise a real-world situation in the language of mathematics; b) to interpret mathematical results in the real-world situation.	Processing Fermi problems require students a) to divide given tasks into manageable subtasks; b) to make reasonable estimates even if there is a lack of data.	Processing STE(A)M modelling tasks require students a) to use mathematics to address scientific (or social, political or economical) problems; b) to connect and combine knowledge from different scientific (or social, political or economical) disciplines.
Student learning	In processing modelling tasks, students learn a) to go beyond pure mathematics; b) to apply their knowledge to a real phenomenon (cf. Blum, 2015).	In processing Fermi problems, students learn a) to set up and work with a mathematical model using basic arithmetic; b) to deal with missing data and to research needed data.	In processing STE(A)M modelling tasks, students learn a) to use mathematics to address scientific (or social, political or economical) problems; b) to recognise modelling as a scientific method in STE(A)M disciplines; c) to further their knowledge in the STE(A)M disciplines and connect it to their mathematical knowledge.
Example	*Adele & Taylor Swift in Concert* (see subsection 7.1.1)	*Over-tourism on Rhodes Island* (see subsection 7.1.2)	*Carbon Footprint* (see subsection 7.1.3)

The above given definitions of types of modelling tasks allows a complex STE(A)M modelling task to be transformed into a Fermi problem or a basic modelling task, depending on the circumstances and need of the educational setting in which the task is used. The transformation is done through different acts of scaffolding (provided in the *dim²ension* app), for example, in terms of providing data, discipline-derived structure (basic modelling tasks) or asking focused questions aimed at revealing relationships and structures (Fermi problems). This idea is illustrated in Figure 7.2.

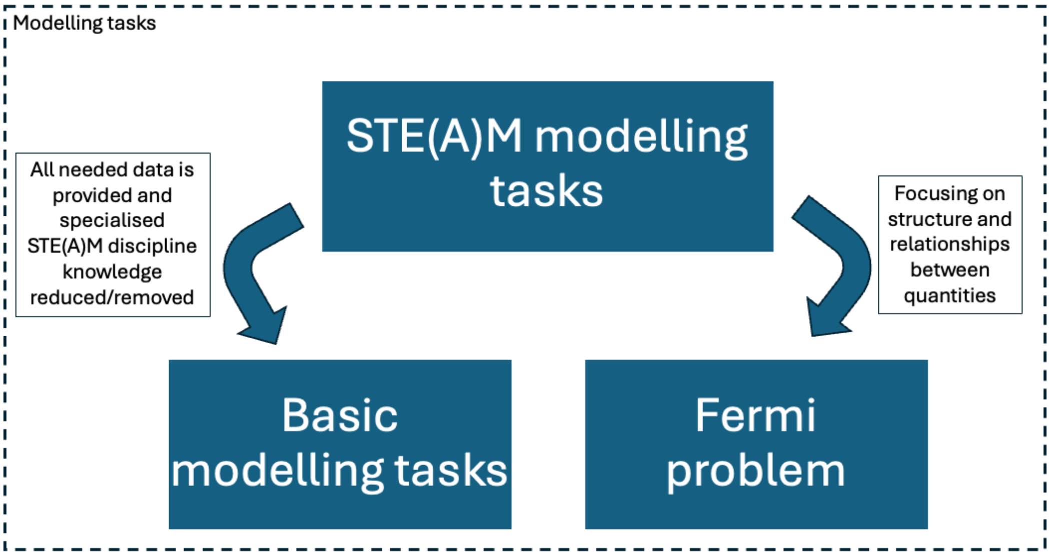

Figure 7.2. The three types of modelling tasks in the *dim²ension* project and possible scaffolding acts to transform and morph STE(A)M modelling tasks into less complex basic modelling tasks or Fermi problems

Example of transforming a task: *Tree Trunk*
To illustrate how a task might be transformed based on different types of scaffolding, consider the following general task: *How much carbon is stored in a European red pine trunk?* By slightly altering the formulation of the task and preparing suitable scaffolding questions aligned with the characteristics of the three different task types respectively, as shown in Table 7.2.

Table 7.2. Summary and examples of the three *dim²ension* modelling types of tasks

Task type	Basic Modelling Task	Fermi Problems	STE(A)M Modelling Task
Task formulation	*The European red pine is an important tree in forestry. How much carbon*	*The European red pine is one of the most common trees in Northern Europe. How much carbon is stored*	*How much carbon is stored in a European red pine?*

	is stored in the trunk of a European red pine?	*in the trunk of all the European red pines in the Swedish National Park Norra Kvill?*	
Scaffold	• Focus on one aspect, namely the tree trunk. • Provide data on the storage of carbon in cellulose or the uptake of carbon in the trunk by photosynthesis.	• Focus on estimation, namely the size of the national park and the ratio of red pine trees in the National Park. • Provide data on the storage of carbon in cellulose or the uptake of carbon in the trunk by photosynthesis.	• Biological knowledge of red pines: How much of the mass of a European red pine is made up of trunk, roots and leaves. • Biological and chemical knowledge of wood: How much carbon is stored in cellulose? • Chemical knowledge of the process of photosynthesis: How much carbon dioxide is absorbed by the carbon stored in the tree trunk?

7.2 Additional task dimensions

As described above, three types of mathematical modelling tasks are distinguished in the *dim^{2}ension* project, namely *basic modelling tasks*, *Fermi problems* and *STE(A)M modelling tasks*. What all three types of tasks have in common is that in the din^{2}ension project all the tasks designed are related to the context of sustainability.

7.2.1 SDG dimension

On the one hand, the link to the SDGs is a straightforward aspect of the task – the tasks are more or less naturally linked to one or more SDGs. However, we still need to define in what way a task addresses a particular SGDs, i.e. which specific learning goals are to be achieved with regards to the SDG. The United Nations (UNESCO, 2017) define three central types of learning objectives for each SDG: cognitive, socio-emotional and behavioural learning goals (see subsection 3.1.2). These three objectives serve as one dimension of the *dim^{2}ension* task classification framework. Each type of modelling task (see 4.1.1) can focus on one or more domains of learning objectives.

- **Cognitive domain:** The task supports the students' understanding of mathematical modelling as an interdisciplinary method for analysing issues of sustainable development. While working on the task, students need to be able to reflect on the quality and limitations of mathematical models for describing and analysing global and local development (Siller et al., 2025a). They apply mathematical models and use their mathematical knowledge and skills to gain deeper insights into the SGDs and the challenges for achieving them. Thereby they experience mathematics as an enhancement of their day-to-day thinking (Heymann, 1996).

- **Socio-emotional domain:** The task requires students to critically question existing models on development issues of globalisation and use results to build consensus with others. They evaluate products, results and their own models of mathematical considerations in an interdisciplinary context and communicate their opinions (Siller et al., 2025a). In doing so, they get the chance to question their own attitudes using mathematics.
- **Behavioural domain:** Students apply concepts and methods of mathematics as tools in individual, social, economic, ecological and political contexts in an interdisciplinary manner and in dealing with statements by experts. They use mathematical concepts and procedures to generate models for global change in a reflective manner and realise and evaluate the consequences of their own actions with regard to the goals of sustainable development with the help of mathematical methods (Siller et al., 2025a).

7.2.2 Complexity dimension

Thinking in (competency) levels is not unusual in everyday school life; rather, curricula and teaching materials are based on this perspective (see, for example, Kiper et al., 2004). There are different frameworks specifying (competency) levels for mathematical activities, among others suggested by well-known international studies such as TIMSS (Mullis et al., 2023). This work is based on the O-M-A competency level model (Siller et al., 2015) as it (1) identifies levels of complexity specifically for modelling tasks and (2) has been shown to be backed by empirical evidence, i.e. the four suggested complexity levels are empirically differentiable (Siller et al., 2015).

The O-M-A model is based on an activity-theoretical background (cf. Lompscher, 1985), including four stages:

1. Performing an action by unreflective imitation (level 1)
2. Performing an action by following instructions (level 2)
3. Performing an action by insight (level 3)
4. Independent process control (level 4)

Level 1 then corresponds to an elementary or schematic pattern orientation. Level 2 is differentiated from level 1 by the complexity of the requirements of the task. Level 3 achieves a so-called field orientation, which is also shown, for example, by the fact that learners are able to generate their own examples. The activity theory approach makes clear that a higher level cannot be reached without first mastering the previous level. However, it is not possible to deduce an individual's level from their specific task processing, since, for example, automation processes resulting from training make it possible to process seemingly difficult tasks at the level of pattern orientation. This is where the demand for high-quality task design becomes crucial. Specifically for modelling tasks, we concretise the following four levels of task complexity (Siller et al., 2015). This is shown in Table 7.3.

Table 7.3. The four suggested complexity levels are empirically differentiable (Siller et al., 2015)

Level 1 **Identification**	Switching representations between context and mathematical representation Using familiar and directly recognizable standard models to describe a given situation with appropriate decision making
Level 2 **Realisation**	Description of the given situation using standard mathematical models or mathematical relationships Recognizing and setting general framework conditions for the use of standard mathematical models
Level 3 **Application**	Applying standard models to novel situations Finding a fit between a suitable mathematical model and a real situation
Level 4 **Complex independent modelling**	Complex modelling of a given situation Reflection on the solution variants or the choice of model and assessment of the accuracy or appropriateness of the underlying solution methods

The four levels show that the change from context to mathematics is taken up again and again. Levels 1 to 3 correspond to pattern or field orientation and include a so-called atomistic view of modelling, which focuses on sub-competencies of the modelling process (cf. Siller & Vorhölter, 2025). Level 4 is significantly more complex and addresses the holistic approach (cf. Siller & Vorhölter, 2025), which essentially involves nothing more than tackling a modelling problem as completely as possible, starting from a real-life situation and using mathematical methods.

We illustrate these levels referring to crude oil consumption in Table 7.4. The overall goal of the task is to determine how long the earth's crude oil reserves will still last based on past annual consumption figures. The precise task description is different depending on the level. For each level, the task description starts as follows: *Reports estimate that by the end of 2023, there were around 204.892.4 million tonnes of crude oil left on the earth. How many years this will last depends on the annual consumption of crude oil.*

Table 7.4. The four suggested complexity levels are empirically differentiable (Siller et al., 2015) applied to the crude oil task

Level 1	*In 1970, the annual global crude oil consumption was at 2243.5 million tonnes, while in 2023, it had already risen to 4530.5 million tonnes per year. If we assume this trend continues linearly, how long will the crude oil still last?*
Level 2	*In the spreadsheet file, you find annual consumption values from*

	1970 to 2023. Drawing on the annual consumption figures from 1970 and 2023, use a suitable mathematical function to describe the development of the annual consumption values over time. Justify your choice and determine how long the crude oil will still last if the development of annual consumption values continues as in the past.
Level 3	*In the spreadsheet file, you find annual consumption values from 1970 to 2023. Use a mathematical model to estimate how long the crude oil will last and justify your model choice.*
Level 4	*In the spreadsheet you find the annual consumption values globally as well as for several countries (e.g., Germany, US, China, India). Considering various scenarios for future developments, draw up several forecasts for how long the crude oil on the earth will still last.*

At **level 1**, learners are required to apply a familiar linear function based on two explicitly provided data points, translating the given information into a mathematical representation. At **level 2**, the task becomes more complex, as multiple data points are given. Learners must justify their model choice by identifying relevant conditions, though the use of the first and last data point to construct a function remains a standard mathematical approach. At **level 3**, model selection is left to the learner, who must independently determine a suitable model (e.g., two-point linear function or linear regression) and apply it appropriately to the problem context. At **level 4**, the complexity increases significantly as learners analyse developments across different countries, requiring the application and comparison of various linear and non-linear models to make critical, mathematically sound forecasts. These four complexity levels constitute a third dimension of the task classification framework.

7.3 *dim²ension* task classification framework

In the previous three (sub)sections, three characteristics of the task have been presented:

- *Task Type:* In section 7.1., the three task types addressed by the *dim²ension* project were introduced. These are the so-called *basic modelling tasks*, *Fermi problems*, and *STE(A)M modelling tasks*.
- *SDG Integration:* In subsection 7.2.1, the integration of the Sustainable Development Goals was defined by three levels, namely the *cognitive domain*, the *socio-emotional domain*, and *the behavioural domain*.
- *Model Complexity:* In subsection 7.2.2, the complexity of the task in terms of the modelling processes to be carried out in order to solve the task within a particular solution strategy. Here, four levels were identified referred to as *identification*, *realisation*, *application*, as well as *complex independent modelling*.

In the *dim²ension* project, we use these three task characteristics to set up a *task classification framework* for mathematical modelling tasks dealing with sustainability issues. It is illustrated in Figure 7.2.

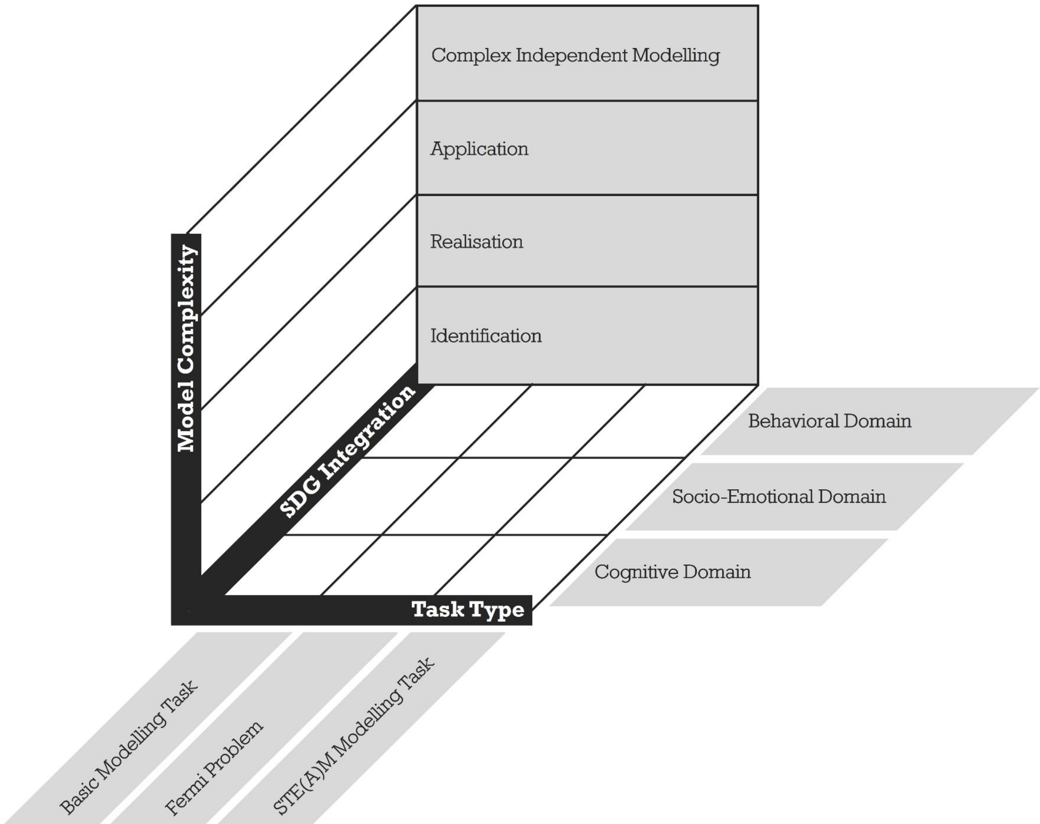

Figure 7.2. The *dim²ension* project's task classification framework

In order to clarify this framework and to illustrate its application, consider the following example task: *How many litres of fresh water do all inhabitants of your city use per day?*

Task Type: Given the characteristics of three task types presented in section 7.1, this task is considered a *Fermi problem* because a sequence of reasonable estimates and educated guesses calculates orders of magnitude (cf. Ärlebäck & Albarracín, 2022).

SDG Integration: The task is related to "SDG 6 - Clean Water and Sanitation". It requires students to calculate the amount of fresh water used in their city. However, it does not directly force students to reflect on their own fresh water consumption (socio-emotional domain), nor does it include a call to action in the sense of working out possibilities to reduce the city's fresh water consumption (behavioural domain). On the contrary, as the tasks focused on building a descriptive model, it is assigned to the *cognitive domain*.

Model Complexity: This task requires students to identify some important information. For example, you could estimate the fresh water consumption of a city by multiplying the fresh water consumption per household by the number of households in the city. But how do you estimate the freshwater consumption of an average household? This could be done by estimating the freshwater consumption of everyday activities such as cooking, cleaning and washing. Following this approach, students have to build a novel model consisting of basic arithmetic operations based on several estimates. The model complexity is therefore assigned to the *application level*.

In short, the three dimensions of *Task Type*, *SDG Integration*, and *Model Complexity* provide the **what** in the sense of what are the desired structural characteristics of tasks. In the following, we address **how** to ensure these tasks foster meaningful learning. To do so, we introduce several task design criteria to operationalise this.. The interplay between these two layers is essential for creating tasks that are not only well-classified but also educationally robust.

7.4 *dim²ension* task design criteria

The *dim²ension* project is developing an open database of mathematical modelling tasks. In particular, all tasks created within the *dim²ension* project have three characteristics, according to our normative approach. These are:

Firstly, *dim²ension* tasks treat relevant and real-world authentic contexts, strongly linked to at least sustainability issues (see Chapter 2). Secondly, they require novel solution strategies, i.e. the set-up of innovative mathematical models. Thirdly, they involve learners reflecting on the content, their learning and its impact on their lives and the world.

To ensure these three aims, we define six normative task criteria. All of them are considered as mandatory for *dim²ension* tasks. The criteria are firstly listed in Table 7.5, before being justified based on theoretical considerations.

Table 7.5. The *dim²ension* project's task design criteria (6 out of 7 criteria)

Relevance	The task illustrates the relevance of learning mathematics for everyday life, particularly by a strong link to European and global sustainability issues. Mathematics is not only perceived as a way of solving a given problem, but also as a way of understanding the world.
Authenticity	The task is based on an authentic task context. This includes accurately reflecting non-mathematical

	aspects of the real situation as well as accurately representing the application of mathematics in that situation. This may also include credible copies of real-world situations, e.g. simulations.
Openness	The task allows for multiple solution strategies as well as for approaches at different educational levels.
Critical Thinking	The task promotes critical thinking as it encourages students to question, analyse and draw conclusions about sustainability issues (based on their modelling process).
Interdisciplinary Links	The assignment incorporates concepts from other STE(A)M subjects to demonstrate the multifaceted nature of sustainability and the central role of modelling in such endeavours.
Problem Posing	The task opens up a rich field in which further questions related to the topic can be addressed.

The first task criterion is **relevance**. As Blum and Niss (1991) state, mathematical modelling aims to apply mathematics to interesting and challenging real-world situations. Here, the modelling process demonstrates the value of mathematics in describing, analysing and evaluating real-world situations. Subsequently, mathematical modelling can be seen not only as a way *of problem handling*, but also as a *way of understanding the world* - to visions of mathematical modelling that are actually present in students' perceptions of mathematical modelling (Brown & Stillman, 2016). Furthermore, in contrast to embedded problems or word problems, relevant tasks deal with a real and factual problem from the real world (Greefrath et al., 2017). This has a strong impact on the processing of the task: Relevant tasks, for example, in terms of dealing with real data and in-depth reflection on the given context (cf. Greefrath et al., 2017).

The second task criterion is the **authenticity** of the task context. According to Vos (2011, 2015), the task is classified as authentic if it presents a real-world context that can be certified as genuine and original by stakeholders in the field. Following this definition, the authenticity of a task can be judged independently of the personal relevance and meaning given to the particular topic (Vos, 2011, 2015). According to this definition, simulated, virtual, and augmented worlds are not authentic because they are (possibly simplistic) copies of the real-world context (cf. Vos, 2011, 2015). However, given the importance of simulations in mathematical modelling (Cevikbas et al., 2023; Siller et al., 2023a), we consider simulations as credible copies of situations (Carreira & Baioa, 2018), which can therefore provide authentic learning situations. In this view, we follow Greefrath et al. (2017) who state: “Authenticity means both the authenticity of the extra-mathematical context and of the application of mathematics in this particular situation. The extra-mathematical context must be real and not be specially designed for a certain arithmetical problem. The application of mathematics in this situation must also be reasonable and realistic and should not just be used in mathematics lessons” (Greefrath et al., 2017, p. 934).

As a third task criterion, we define the **openness** of mathematical modelling tasks. Open tasks have either a vague initial state, a vague intermediate state or a vague goal state, i.e. they lack a clear definition of their solution spaces (Schukajlow et al., 2023; Simon, 1973). In addition, open tasks can be solved by different solution strategies, i.e., different mathematical models, and thus can have multiple mathematical results (Klavir, & Hershkovitz, 2008; Schukajlow et al., 2023). Because of this openness, the same mathematical modelling tasks can often be solved at different educational levels (cf. Jablonski et al., in press).

The fourth task criterion is the promotion of **critical thinking**. Critical thinking in mathematical modelling aims to "recognize, understand, analyse and assess representative examples of actual uses of mathematics, including (suggested) solutions to socially significant problems" (Blum & Niss, 1991, p. 43). This statement is still supported today: The promotion of responsible citizenship (Kaiser, 2020), especially in the face of global challenges such as climate change (cf. Geiger, 2024; Steffensen, 2020), is commonly described as a major benefit of mathematical modelling.

As a fifth task criterion, we define **interdisciplinary links** between the mathematical task and other STEM subjects as well as the arts, e.g. politics or economics. The linking of STE(A)M subjects opens the field for relevant and authentic task contexts as well as for critical thinking (Goos et al., 2023; Maas et al., 2019; Siller et al., 2024c). Furthermore, in the light of the United Nations' (2015) SDGs, the integration of STE(A)M subjects in mathematical modelling allows global challenges and sustainability issues to be described, analysed and evaluated using mathematics (cf. Goos et al., 2023; Maas et al., 2019; Siller et al., 2024a).

The sixth task criterion is **problem posing**. Problem posing is defined as "both the generation of new problems and the re-formulation of given problems" (Silver, 1994, p. 19). Regarding mathematical modelling, Hansen and Hana (2015) highlight the role of problem posing in going beyond the given task: By raising further questions and answering them mathematically, the students can, for example, express and justify expectations or test conjectures through "What if...?" statements. Particularly in the context of sustainability, problem posing is considered of utmost importance (Siller et al., 2024a): Asking and answering additional questions helps students to cope with the high complexity of sustainability issues. Therefore, by providing relevant, authentic and open-ended modelling tasks and by stimulating students' critical thinking, we encourage students to ask further questions about the sustainability issue addressed by the task. In doing so, we aim to stimulate a deeper understanding and examination of the context of the task (cf. Hartmann et al., 2023).

Finally, in our normative approach, we also set a desirable task aspect, shown in Table 7.6.

Table 7.6. The *dim²ension* project's task design criteria (the 7th criteria)

Interactive Elements	The task uses digital tools, simulations and hands-on activities to make learning dynamic and engaging.

We suggest the use of **interactive elements** to make the task dynamic and engaging. This can be done through digital tools such as dynamic GeoGebra applets (Greefrath & Siller, 2018), as well as simulated, virtual and augmented worlds (Siller et al., 2023a). In addition, it can also include hands-on activities (cf. Carreira & Baioa, 2018).

7.4.1 Connecting task design criteria and task dimensions

As introduced in sections 7.1 and 7.2, the dim²ension task framework consists of three dimensions, namely Task Type, SDG Integration, and Model Complexity. These three dimensions form the structural backbone for classifying modelling tasks. However, classification alone does not guarantee pedagogical quality of the tasks that get implemented in the classroom. To translate these dimensions into actionable and attainable design principles, one needs to articulate how these three dimensions inform the *normative design criteria* on which to base the three different types of modelling tasks. To do this, at a conceptual level, the logic is as follows:

Task Type determines the scope of mathematical and interdisciplinary engagement. These in turn set minimum standards for **authenticity** (*is the non-mathematical context credible?*), **interdisciplinarity** (*which STE(A)M domains must be meaningfully invoked?*), and **openness** (*how many viable solution paths are available*). For example, STE(A)M modelling tasks inherently demand stronger interdisciplinary integration and thus must satisfy authenticity beyond mathematics.

SDG Integration aligns with the criteria of **Relevance** and **Critical Thinking**. Tasks mapped to cognitive, socio-emotional, and behavioural domains should embed opportunities for reflection and action, ensuring that sustainability is not treated as a superficial context but as a driver of reasoning and decision-making. Thus, for all SDG integration levels, the SDG is seen as the driver of the task. Depending on the SDG integration level, the nature of reflection and action the task should provoke is determined (analysis of claims, values discussion, and tangible behaviour/decision-making).

Model Complexity informs the depth of reasoning and the degree of scaffolding required, which connects to **Openness** *(multiple strategies, parameter choices)* and **Problem Posing** *("What if ...?"-extensions)*. Higher complexity levels (e.g., Level 4: independent modelling) necessitate tasks that allow multiple solution paths and encourage learners to generate new questions.

As argued and presented above, the classification dimensions introduced in chapter 7.3 provide the structural profile of a modelling task (what type of task it is, how it engages with the SDGs, and at which level of modelling complexity it operates). The seven design criteria just discussed on the other hand stipulate the pedagogical quality filters that any task must satisfy to be educationally robust. In practice, each dimension in chapter 7.3 to various extent triggers particularly some specific expectations in terms of the seven design criteria, while other criteria are still graspable but less explicitly addressed. . This is illustrated in the following by a task example.

7.4.2 Illustrating connections: Over-tourism on Rhodes Island

The task described in subsection 7.1.1, *Over-tourism on Rhodes Island*, in terms of its **Task Type** is clearly a *Basic Modelling Task*, as students are asked to apply a simple, pre-defined mathematical model (the Visitor-to-Resident Ratio) to a real-world situation without requiring advanced interdisciplinary computations. This aligns with the definition in subsection 7.1.1, where the emphasis is on using mathematics to interpret a concrete phenomenon rather than constructing a novel model from scratch. Second, the task is strongly anchored in **SDG Integration**, specifically SDG 12 (*Responsible Production and Consumption*). The context of tourism pressure on Rhodes Island directly addresses sustainability concerns, and the formulation invites learners to reflect on infrastructure and community impacts. This means the task primarily targets cognitive learning objectives (understanding ratios and sustainability indicators) but also hints at behavioural dimensions by asking students to propose solutions. Third, the **Model Complexity** corresponds to Level 2 (*Realisation*) in the O-M-A framework. Learners are intended to use or reproduce a model; they must apply the ratio to a novel context, interpret the results, and reason about consequences, which requires insight and decision-making beyond procedural execution to make use of standard mathematical knowledge. This is summarized in Table 7.7.

In terms of adhering to the design criteria the **Relevance** of the *Over-tourism on Rhodes Island* task is evident: the problem addresses a pressing sustainability issue—over-tourism—making mathematics meaningful in a societal context. **Authenticity** is achieved by situating the task in a real geographical setting and using a widely recognized indicator, ensuring that both the context and the mathematical application reflect genuine practice. The principle of **Openness** is supported through interpretive flexibility: while the ratio formula is fixed, students can explore seasonal variations, alternative thresholds, or additional capacity indicators, allowing multiple solution paths. **Critical Thinking** is embedded in the requirement to justify assumptions (e.g., what constitutes "unsustainable" tourism) and to evaluate the limitations of a single indicator, fostering analytical reasoning. **Interdisciplinary Links** emerge naturally as the discussion extends beyond mathematics to geography, economics, and environmental studies, particularly when considering infrastructure and resource constraints. The task also promotes **Problem Posing**, as learners are encouraged to propose strategies for sustainable tourism, which necessitates generating new questions such as *What if visitor numbers decrease by 20%?* or *What if water supply is expanded?* These considerations are summarized in Table 7.7.

Table 7.7. A summary of the connection between the *dim²ension* framework task dimension and the project's task design criteria in the case of the basic modelling tasks *Over-tourism on Rhodes Island*

Task dimensions	Adherence to the task design criteria
Task Type: Basic	Requires strong **Authenticity** (real data) and **Relevance**; low **Openness** for model and moderate **Openness** for interpretation.
SDG Integration: SDG 12	Drives **Relevance** and **Critical Thinking**; supports cognitive goals and subordinately **b**ehavioural goals via solution proposals.
Complexity: Level 2	Necessitates **Problem Posing** and scaffolds for interpretation, not just computation.

7.5 *dim²ension* web portal and app

In the previous sections, a task classification system and task criteria for the *dim²ension* project were derived. This section outlines how these theoretical considerations can be brought to life. More specifically, based on the project status as of June 2025, we outline how the user's workflow is planned. A full description can be found in Barlovits et al. (submitted).

The *dim²ension* system is a digital tool to support mathematical modelling. It consists of a web portal and a mobile app. While the web portal is designed as a workspace for teachers, the app is developed as a workspace for students. Roughly speaking, the mobile app serves as a guide for students to process modelling tasks. The web portal, on the other hand, allows all kinds of management functions, from creating or selecting modelling tasks to evaluating students' results.

Before we describe the most important functionalities of the web portal and the mobile app, we firstly sketch how a mathematical modelling task is structured from a technical point of view in *dim²ension*.

7.5.1 Task structure and work modes

In technical terms, a *dim²ension* task consists of the following elements:

- a task title,
- a task description, i.e., the task formulation,
- a visualisation of the given situation, namely a picture, a GIF or a video,
- at least one scaffolded approach, i.e., a pre-guided solution strategy including a sample solution,
- resources such as external links, applets or files (optional),
- meta information about the task and the related approaches, such as the task type, the SDG integration level or the model complexity.

To work on a task in *dim²ension*, students can follow one of two working modes: They can either work on the task in the *blank canvas mode* as a free working mode, or they can work in the *scaffolding mode*, which helps students to guide and structure their working process. The latter mode is especially designed for beginners and is used to learn how to model mathematically.

In both working modes, we follow the idea of *modelling through questions*: Students either ask themselves support questions (*blank canvas mode*) or follow predefined support questions (*scaffolding mode*).

Scaffolding Mode:
The scaffolding mode provides students with step-by-step guidance, including support questions, hints, and resources, to help them navigate the modelling process. The blank canvas mode, on the other hand, encourages students to ask their own support questions and develop their own solution strategies. Both modes aim to foster cognitive activation, critical thinking, and creativity.

In the scaffolding mode, the students can access a pre-defined approach to process a particular task. Each task approach consists of a number of support questions which are presented in a meaningful order to complete the task.

This idea will be illustrated using the Fermi problem *Adele & Taylor Swift in Concert* from subsection 7.1.2. The task asks students to calculate the additional CO_2 emissions in tonnes for fan mobility for a single concert in an Adele tour in Munich, Germany, compared to a Taylor Swift concert in Gelsenkirchen, Germany. One approach that could help students decompose this task into the following support questions is presented in Table 7.8.

Table 7.8. The *dim²ension* scaffolding mode, illustrated with the Fermi problem *Adele & Taylor Swift in Concert*

Full Task	Estimate the amount of additional CO2 [in tonnes] for fan mobility that will be emitted for a single concert in an Adele concert series compared to a Taylor Swift world tour.

Support Question 1	What are the average CO_2 emissions per fan depending on where they live?
Support Question 2	What are the average CO_2 emissions per fan depending on where they live and their chosen mode of transport?
Support Question 3.1	How many of the 73,000 fans at the Adele concert came from Germany, other European countries, and countries outside Europe? How many attended the Taylor Swift concert?
Support Question 3.2	What mode of transportation did fans from Germany, other European countries, and countries outside Europe choose?

The scaffolding idea is to provide stepped support without revealing the mathematical model. If the students need advice, then a support question which is slightly easier to answer than the given task is presented. If students still need advice, the next easier support question is presented, and so on. In this way, the system provides scaffolding and guidance, while constantly challenging students to engage in cognitive activation, critical thinking and mathematical creativity (see subsections 5.2 - 5.4).

Each support-question can be supplemented with *hints* and *resources*, including documents, links and applets. Moreover, an explanation can be provided to students to highlight the relevance of the support question. For *Support Question 2.1* from the above example, such an intention might be:"Assuming an average travel distance for fans from Germany, Europe and outside Europe, we can estimate their CO_2 emissions from travelling to the concert, depending on their chosen means of transport." This core idea of the *dim²ension* system is described in detail by Barlovits et al. (submitted).

Once they have completed the task, students can enter the *numerical results* into the app and upload *materials* such as documents or pictures of their notes. The teacher can then access these materials via the *Digital Classroom* (see subsection 7.5.3). Finally, the students can answer *reflection questions*, which encourage them to think critically about the given ESD context and/or their modelling process.

Blank Canvas Mode:

In the blank canvas mode, the *dim²ension* app poses only strategic hints to help students to structure their solution process, such as "Make a sketch" or "Look for the data you need and, if necessary, make assumptions" (cf. Schukajlow et al., 2015). However, no guidance is provided at a context-specific level.

Students are asked to ask themselves meaningful support questions to make the task manageable. The support questions posed are enriched by the intention of the support question, the student's response to the support question, and additional resources if needed.

By connecting the support questions and working through them step by step, the whole picture emerges and students find their own way of processing the given task.

7.5.2 *dim²ension* web portal

There are two ways to prepare a mathematical modelling activity within *dim²ension*: Teachers can either select tasks from the open database created within the *dim²ension* project. Or they can decide to create their own mathematical modelling task. Both ways of using the *dim²ension* web portal are described below.

Task Selection: In the web portal, teachers find a database of all *dim²ension* modelling tasks. Teachers can select the tasks they consider appropriate for their students by filtering, e.g. by task type, SDG integration level or model complexity.

If they find a promising task in the database, teachers can access the task in the web portal and view the task formulation, related task resources (such as external links, applets or files), and all related task approaches.

Task Creation: The web portal provides teachers with a step-by-step form to help them create their own tasks. Firstly, the task context is set up. The second step involves defining different approaches, which are then enriched by support questions in the third step.

In the first step, the task author provides a formulation and an illustration of the task, i.e. an image depicting the real situation. The task can be enriched with further resources, e.g. an external link to an information text on Wikipedia, a GeoGebra applet or a file such as a spreadsheet with given data. In addition, meta information about the tasks is defined, such as the task type or the SDG integration level.

In a second step, at least one approach for the scaffolding mode is defined. As described above, one approach consists of various support questions, which are enriched by assistance and answering elements. Also, meta information about the approach is defined, e.g., on the model complexity. In a third step, one can create the support questions.

7.5.3 *dim²ension* app & Digital Classroom

Students can access tasks in the mobile app in one of three ways: via a task-related code, from a task list provided, or via the so-called *Digital Classroom*. This feature of the *dim²ension* web portal is a virtual representation of one learning group. Teachers can manage their learning group by assigning tasks to their students in the web portal. Students can access the *Digital Classroom* by scanning a specific QR code via the *dim²ension* mobile app.

After a student enters a Digital Classroom via the *dim²ension* mobile app, all mathematical modelling tasks assigned by the teacher are automatically displayed. Students can access the task and work either in the *blank canvas mode* or the *scaffolding mode*.

In both working modes, students can upload their numerical result, if appropriate, as well as a documentation of their solution process, e.g., a photo of their notes. This information is stored in the Digital Classroom and can be retrieved by the relevant students and their teacher. This allows for effective evaluation and debriefing of the modelling activity. In line with GDPR requirements, the Digital Classroom feature also allows teachers to monitor student progress in real time, for example, in terms of time taken, use of hints or solutions entered. This data is summarised by an automated assessment.

7.5.4 Key features

In view of the major challenges perceived by teachers (Borromeo Ferri & Blum, 2010; Jablonski et al., submitted, Wess et al., 2021) and learners (Blum, 2015; Galbraith & Stillman, 2006; Klock & Siller, 2020) in mathematical modelling, the dim²ension system aims to facilitate the implementation of mathematical modelling activities in educational practice.

The *dim²ension* project (Barlovits et al., submitted) accompanies teachers from the preparation to the evaluation of mathematical modelling activities:

- Preparation: In the *dim²ension* web portal, teachers can either select tasks from an open database or create their own tasks,
- Implementation & Management: To carry out modelling activities with *dim²ension*, teachers simply need to assign the selected task to their learning group's Digital Classroom.
- Monitoring & Evaluation: In line with GDPR requirements, teachers can monitor students' progress on tasks. All students' results are stored in the Digital Classroom and are visible to the teacher and the relevant students.

Students are supported by the system as follows: Based on their prior knowledge of mathematical modelling, they can choose to challenge themselves and work independently on the given task *(blank canvas mode)*, receiving only strategic hints. Or they can choose to learn how to model and receive guidance and support as needed *(scaffolding mode)*. In both modes, the mobile app helps students to easily access related resources, structure their solution process and document their results.

By providing the *dim²ension* system, we are exploiting a wide range of described potentials of digital tools to support mathematical modelling (cf. Cevikbas et al., 2023; Siller et al., 2023a). In doing so, we aim to support students in building, extending and deepening their mathematical modelling skills in order to address global sustainability issues through mathematics.

8. Conclusion and outlook

This theoretical white paper has laid the foundation for the *dim²ension* project by articulating a comprehensive and interdisciplinary framework for integrating digital-supported mathematical modelling into Education for Sustainable Development (ESD) across European contexts. Grounded in robust theoretical perspectives and empirical insights, the document has systematically defined and contextualized the core concepts of models, mathematical models, and modelling, while also exploring their pedagogical implications in mathematics education.

The paper has emphasized the transformative potential of mathematical modelling as a bridge between abstract mathematical thinking and real-world problem-solving. By embedding modelling within the broader educational goals of the United Nations' (2015) Sustainable Development Goals (SDGs), the *dim²ension* project positions mathematics as a critical tool for fostering responsible citizenship, critical thinking, and interdisciplinary understanding. The integration of digital tools – particularly mobile learning technologies and web-based platforms – further enhances the accessibility, interactivity, and contextual relevance of modelling tasks, enabling learners to engage with complex global challenges in meaningful and situated ways.

A central contribution of this white paper is the development of a task classification and design framework that aligns modelling activities with three key dimensions: task type (basic modelling tasks, Fermi problems, and STE(A)M modelling tasks), SDG integration (cognitive, socio-emotional, and behavioural learning objectives), and task complexity (based on the O-M-A model). This framework not only supports the systematic design and evaluation of modelling tasks but also ensures their alignment with pedagogical goals and learner needs. The *dim²ension* system – comprising a mobile app and web portal – operationalizes this framework, offering educators and students a flexible, scaffolded environment for engaging with modelling tasks in both classroom and outdoor settings.

Looking ahead, the *dim²ension* project aims to further refine and expand its theoretical and practical contributions. Future work will focus on technical, material-based, educational and research-based project outcomes:

- **Technical outcomes:** Iterative development of the *dim²ension* digital tools, informed by user experience studies and technological advancements.
- **Material-based outcomes:** Expansion of the task database, with a focus on increasing diversity in SDG topics, cultural contexts, and disciplinary integration.
- **Educational outcomes:** Professional development initiatives to support teachers in adopting modelling-based pedagogies and leveraging digital tools effectively.
- **Research-based outcomes I:** Empirical validation of task classification framework and design criteria as well as of technical, material-based and

educational outcomes through classroom implementation and teacher feedback.
- **Research-based outcomes II:** Research dissemination through academic publications, policy briefs, and international collaborations to promote the uptake of the *dim²ension* approach across educational systems.

In conclusion, the *dim²ension* project represents a significant step toward reimagining mathematics education as a dynamic, interdisciplinary, and socially responsive practice. By equipping learners with the tools to model, analyse, and act upon real-world challenges, the project contributes to the cultivation of mathematically literate and sustainability-conscious citizens for the 21st century.

References

Ahlfeldt, S., Mehta, S., & Sellnow, T. (2005). Measurement and analysis of student engagement in university classes where varying levels of PBL methods of instruction are in use. *Higher Education Research & Development*, *24*(1), 5–20.

Akerele, O. (2017). Software process simulation modelling for agile cloud software development projects: Techniques and applications. In *Strategic Engineering for Cloud Computing and Big Data Analytics* (pp. 119–139). Springer. https://doi.org/10.1007/978-3-319-52491-7_7

Albarracín, L., & Ärlebäck, J. B. (2019). Characterising mathematical activities promoted by Fermi problems. *For the Learning of Mathematics, 39*(3), 10–13. https://ddd.uab.cat/record/296776

Albarracín, L., & Gorgorió, N. (2014). Devising a plan to solve Fermi problems involving large numbers. *Educational Studies in Mathematics, 86*(1), 79–96. https://doi.org/10.1007/s10649-013-9528-9

Anderson, J., & White, P. (2004). Problem solving in learning and teaching mathematics. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 127–150). Post Pressed.

Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Montana Mathematics Enthusiast, 6*(3), 331–364. https://doi.org/10.54870/1551-3440.1157

Ärlebäck, J. B., & Albarracín, L. (2019). The use and potential of Fermi problems in the STEM disciplines to support the development of 21st century competencies. *ZDM – Mathematics Education, 51*(7), 979–990. https://doi.org/10.1007/s11858-019-01075-3

Ärlebäck, J. B., & Albarracín, L. (2022). FPAT – a framework for facilitating the teaching and learning of using Fermi problems originating in Mathematics Education Research. *The Physics Teacher*, *60*(7), 569–571. https://doi.org/10.1119/5.0040017

Ärlebäck, J. B., & Albarracín, L. (2023). Fermi problems as a hub for task design in mathematics and STEM education. *Teaching Mathematics and its Applications: An International Journal of the IMA, 43*(1), 25–37. https://doi.org/10.1093/teamat/hrad002

Ärlebäck, J. B., & Bergsten, C. (2010). On the use of realistic Fermi problems in introducing mathematical modelling in upper secondary mathematics. In R. Lesh, P. L. Galbraith, W. Blum, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies: ICTMA 13* (pp. 597–609). Springer.

Ärlebäck, J. B., Carreira, S., Greefrath, G., Eyrich Jessen, B., Lacek, Y., & Vorhölter, K. (2023). Introduction to the papers of TWG6: Application and modelling. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 1095–1102). Alfréd Rényi Institute of Mathematics and ERME.

Ärlebäck, J. B., Sevinc, S., & Albarracín, L. (accepted). The wheel of mathematical modelling: Seeking a new perspective. Paper to appear in the proceedings of CERME14.

Avelar, A. B. A., da Silva Oliveira, K. D., & Farina, M. C. (2023). The integration of the Sustainable Development Goals into curricula, research and partnerships in higher education. *International Review of Education.* https://doi.org/10.1007/s11159-023-10013-1

Avigad, J. (2020). The mechanization of mathematics. *Notices of the American Mathematical Society*, *67*(1), 30–39. https://doi.org/10.1090/noti202001

Aydogan Yenmez, A., Erbas, A. K., Alacaci, C., Cakiroglu, E., & Cetinkaya, B. (2017). Evolution of mathematics teachers' pedagogical knowledge when they are teaching through modeling. *International Journal of Education in Mathematics, Science and Technology, 5*(4), 317–332. https://doi.org/10.18404/ijemst.296552

Ball, L., Drijvers, P., Ladel, S., Siller, H.-S., Tabach, M., & Vale, C. (Eds.). (2018). *Uses of technology in primary and secondary mathematics education: Tools, Topics and Trends. ICME-13 Monographs*. Springer. https://doi.org/10.1007/978-3-319-76575-4

Baptista, M., Jacinto, H., & Martins, I. (2023). What is a good explanation in integrated STEM education?. *ZDM – Mathematics Education*, *55*(7), 1255–1268.

Barbier, E. B., & Burgess, J. C. (2017). The sustainable development goals and the systems approach to sustainability. *Economics, 11*(1), 2017–2028. https://doi.org/10.5018/economics-ejournal.ja.2017-28

Barke, A., Sodhi, M. S., Thies, C., & Spengler, T. S. (2023). Linking life cycle sustainability assessment and the sustainable development goals – Calculation of goal achievement. *Procedia CIRP, 116*, 618–623. https://doi.org/10.1016/j.procir.2023.02.104

Barlovits, S., Fessakis, G., Ludwig, M., & Kolokytha, A. (2022a). Designing mobile environments for mathematics distance education: The theory-driven development of the ASYMPTOTE system. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 2496–2503). Free University of Bozen-Bolzano & ERME.

Barlovits, S., Caldeira, A., Fesakis, G., Jablonski, S., Koutsomanoli Filippaki, D., Lázaro, C., Ludwig, M., Mammana, M. F., Moura, A., Oehler, D.-X. K., Recio, T., Taranto, E., & Volika, S. (2022b). Adaptive, synchronous, and mobile online education: Developing the ASYMPTOTE learning environment. *Mathematics*, *10*(10), 1628. https://doi.org/10.3390/math10101628

Barlovits, S., Jablonski, S., Oehler, D.-X. K., & Ludwig, M. (2024). How learning platforms support mathematical modelling: An exemplary approach via the ASYMPTOTE system. In E. Faggiano, A. Clark-Wilson, M. Tabach, & H.-G. Weigand (Eds.), *Proceedings of the 17th ERME Topic Conference MEDA 4* (pp. 41–48). University of Bari Aldo Moro.

Barlovits, S., Ludwig, M., Ärlebäck, J. B., Carreira, S. Fesakis, G., Fock, A., Jacinto, H., Julsgård, S., Läufer, T., Lázaro del Pozo, C., Oehler, D.-X. K., Perdikaris, K., Reyes Delgado, J. A., & Siller, H.-S. (submitted). Entering the next dim²ension: How mathematical modelling in the context of sustainability can be supported by a digital resource. In *International Perspectives on the Teaching and Learning of Mathematical Modelling*. Springer.

Barrows, H. S. (1986). A taxonomy of problem-based learning methods. *Medical Education, 20*(6), 481–486. https://doi.org/10.1111/j.1365-2923.1986.tb01386.x

Barrows, H. S. & Tamblyn, R. M. (1980). *Problem-Based Learning. An Approach to Medical Education*. Springer.

Baumert, J., Kunter, M., Blum, W., Klusmann, U., Krauss, S., & Neubrand, M. (2013). Professional competence of teachers, cognitively activating instruction, and the development of Students' Mathematical Literacy (COACTIV): a research program. In M. Kunter, J. Baumert, W. Blum, U. Klusmann, S. Krauss, & M. Neubrand (Eds.), *Cognitive activation in the mathematics classroom and professional competence of teachers* (pp. 1–21). https://doi.org/10.1007/978-1-4614-5149-5_1

Bernárdez-Gómez, A., Martínez, M. J. B., González-Barea, E. M., & Rodríguez-Entrena, M. J. (2024). How are the Sustainable Development Goals being worked within the schools? *Journal of Teacher Education for Sustainability, 26*(2). https://doi.org/10.2478/jtes-2024-0021

Beyer, B. K. (1995). *Critical thinking*. Bloomington, IN: Phi Delta Kappa Educational Foundation.

Black, M. (1962). *Models and Metaphors: Studies in Language and Philosophy*. Cornell University Press.

Blomhøj, M., & Højgaard Jensen, T. (2007). What's all the fuss about competencies? In W. Blum, P. L. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 45–56). Springer.

Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: conceptual clarification and educational planning. *Teaching Mathematics and its Applications*, *22*(3), 123–139.

Bloom, B. S., Engelhart, M. D., Furst, E. J., Hill, W. H., & Krathwohl, D. R. (1956). *Taxonomy of educational objectives: The classification of educational goals. Handbook I: Cognitive domain*. David McKay.

Blum, W. (1985). Anwendungsorientierter Mathematikunterricht in der didaktischen Diskussion [Application-oriented mathematics teaching in didactic discussion]. *Mathematische Semesterberichte*, *32*, 195–232.

Blum, W. (2006). Modellierungsaufgaben im Mathematikunterricht – Herausforderung für Schüler und Lehrer [Modeling tasks in mathematics lessons – a challenge for students and teachers]. In A. Büchter, H. Humenberger, S. Hußmann, & S. Prediger (Eds.), *Realitätsnaher Mathematikunterricht – vom Fach aus und für die Praxis: Festschrift für Hans-Wolfgang Henn zum 60. Geburtstag* (pp. 8–23). Franzbecker.

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do?. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education (ICME12)* (pp. 73–96). Springer. https://doi.org/10.1007/978-3-319-12688-3_9

Blum, W., & Leiß, D. (2005). Modellieren im Unterricht mit der „Tanken"-Aufgabe [Modelling in class with the 'refueling' task]. *Mathematik Lehren*, *128*, 18–21.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical Modelling: Education, Engineering and Economics–ICTMA 12* (pp. 222–231). Horwood Publishing. https://doi.org/10.1533/9780857099419.5.221

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects — State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37–68. https://doi.org/10.1007/BF00302716

Blum, W., Galbraith, P. L., Henn, H., & Niss, M. (Eds.). (2007). *Modelling and applications in mathematics education. The 14th ICMI study*. Springer.

Blum, W. & Borromeo Ferri, R. (2009). Mathematical modelling: can it be taught and learnt?. *Journal of Mathematical Modelling and Application*, *1*(1), 45–58.

Bock, W., Bracke, M., & Kreckler, J. (2015). Taxonomy of modelling tasks. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME 9)* (pp. 821–826). Charles University in Prague, Faculty of Education and ERME. https://hal.science/hal-01287249

Bonotto, C., & Santo, L. D. (2015). On the relationship between problem posing, problem solving, and creativity in the primary school. In B. Sriraman (Ed.), *The interdisciplinary nature of reasoning and problem solving* (pp. 103–123). Springer (pp. 103–123). https://doi.org/10.1007/978-1-4614-6258-3_5

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM – Mathematics Education*, *38*(2), 86-95.

Borromeo Ferri, R. (2007a). Modelling problems from a cognitive perspective. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, engineering and economics: proceedings from the twelfth International Conference on the Teaching of Mathematical Modelling and Applications* (pp. 260-270). Horwood.

Borromeo Ferri, R. (2007b). Personal experiences and extra-mathematical knowledge as an influence factor on modelling routes of pupils. In D. Pitta-Pantazi & G. Philippou (Eds.), *European Research in Mathematics Education V: Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education* (pp. 2080-2089). University of Cyprus.

Borromeo Ferri, R. (2018). *Learning how to teach mathematical modeling in school and teacher education.* Springer. https://doi.org/10.1007/978-3-319-68072-9

Borromeo Ferri, R., & Blum, W. (2010). Mathematical modelling in teacher education–experiences from a modelling seminar. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (CERME 6)* (pp. 2046–2055). INRP.

Bossel, H. (1992). *Modellbildung und Simulation: Konzepte, Verfahren und Modelle zum Verhalten dynamischer Systeme* [Modeling and simulation: Concepts, methods, and models for the behavior of dynamic systems]*.* Vieweg.

Botzer, G. & Yerushalmy, M. (2007). Mobile applications for mobile learning. In *Proceedings of IADIS: International Conference Cognition and Exploratory Learning in the Digital Age (CELDA)* (pp. 7–9).

Brenner, J. E. (2010). The logical process of model-based reasoning. In L. Magnani, W. Carnielli & C. Pizzi (Eds.), *Model-based reasoning in science and technology (pp. 333–358). Springer.* https://doi.org/10.1007/978-3-642-15223-8_19

Brown, J. P., & Stillman, G. A. (2016). Developing the roots of modelling conceptions: ‘Mathematical modelling is the life of the world’. *International Journal of Mathematical Education in Science and Technology*, *48*(3), 353–373. https://doi.org/10.1080/0020739X.2016.1245875

Brown, J. P., Stillman, G. A., Galbraith, P. L., & Ng, K. E. D. (2024). Mathematical modelling of real-world phenomena. In G. A. Stillman, J. P. Brown, & P. L. Galbraith (Eds.), *Research in mathematics education in Australasia 2020–2023* (pp. 141–164). Springer. https://doi.org/10.1007/978-981-99-6043-5_9

Brundtland, G. H. (1987). *Report of the world commission on environment and development: Our common future: Note by the Secretary-General* (A/42/427). United Nations. https://documents.un.org/doc/undoc/gen/n87/184/67/pdf/n8718467.pdf?token=Smrbr3IjHmaj1BgOmm&fe=true

Buchholtz, N. (2021). Modelling and mobile learning with math trails. In F. K. S. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *International perspectives on the teaching and learning of mathematical modelling* (pp. 331–340). Springer. https://doi.org/10.1007/978-3-030-66996-6_28

Buchholtz, N., & Mesrogli, S. (2013). A whole week of modelling–examples and experiences of modelling for students in mathematics education. In G. Stillman, G. Kaiser, W. Blum & J. Brown (Eds.), *Teaching mathematical modelling: connecting to research and practice* (pp. 307–316). Springer.

Bukova-Güzel, E. (2011). An examination of pre-service mathematics teachers' approaches to construct and solve mathematical modelling problems. *Teaching Mathematics and Its Applications, 30*(1), 19–36. https://doi.org/10.1093/teamat/hrq023

Burke, P. F., Kearney, M., Schuck, S., & Aubusson, P. (2021). Improving mobile learning in secondary mathematics and science: Listening to students. *Journal of Computer Assisted Learning*, *38*(1), 137–151. https://doi.org/10.1111/jcal.12596

Cai, J., Kaiser, G., Leikin, R., & Ärlebäck, J. B. (2025). *Relations among problem solving/posing creativity and mathematical modeling.* In T. Ikeda, A. Saeki, V. Geiger, & G. Kaiser (Eds.), *International Horizons in Mathematics Modelling Education. International Perspectives on the Teaching and Learning of Mathematical Modelling* (pp. 333–343). Springer. https://doi.org/10.1007/978-3-031-53533-8_3

Calder, N., Larkin, K., & Sinclair, N. (Eds.). (2018). *Using mobile technologies in the teaching and learning of mathematics* (Vol. 12). Springer. https://doi.org/10.1007/978-3-319-90179-4

Candy, L. (1997). Computers and creativity support: Knowledge, visualisation and collaboration. *Knowledge-Based Systems*, *10*, 3-13. https://doi.org/10.1016/S0950-7051(97)00008-7

Carreira, S., & Baioa, A. M. (2018). Mathematical modelling with hands-on experimental tasks: On the student's sense of credibility. *ZDM – Mathematics Education*, *50*(1), 201–215. https://doi.org/10.1007/s11858-017-0905-1

Carter, A. G., Creedy, D. K., & Sidebotham, M. (2017). Critical thinking evaluation in reflective writing: Development and testing of Carter Assessment of Critical Thinking in Midwifery (Reflection). *Midwifery, 54*, 73–80. https://doi.org/10.1016/j.midw.2017.08.003

Cevikbas, M., Greefrath, G., & Siller, H.-S. (2023). Advantages and challenges of using digital technologies in mathematical modelling education–a descriptive systematic literature review. *Frontiers in Education*, *8*, 1142556. https://doi.org/10.3389/feduc.2023.1142556

Cevikbas, M., Kaiser, G., & Schukajlow, S. (2022). A systematic literature review of the current discussion on mathematical modelling competencies: State-of-the-art developments in conceptualizing, measuring, and fostering. *Educational Studies in Mathematics*, *109*(4), 205–236. https://doi.org/10.1007/s10649-021-10104-6

Chen, Y. (2019). Effect of mobile augmented reality on learning performance, motivation, and math anxiety in a math course. *Journal of Educational Computing Research*, *57*(7), 1825–1845. https://doi.org/10.1177/0735633119854036

Choi, E., Lindquist, R., & Song, Y. (2014). Effects of problem-based learning vs. traditional lecture on Korean nursing students' critical thinking, problem-solving and self-directed learning. *Nurse Education Today*, *34*, 52–56. https://doi.org/10.1016/j.nedt.2013.02.012

Clarke, D., Emanuelsson, J., Jablonka, E., & Mok, I. A. C. (2009). *Making connections: Comparing mathematics classrooms around the world.* Sense Publishers.

Clarke, D., Strømskag, H., Johnson, H. L., Bikner-Ahsbahs, A., & Gardner, K. (2014). Mathematical tasks and the student. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (vol. 1, pp. 117–143). PME.

Confrey J. & Maloney, A. (2007). A theory of mathematical modelling in technological settings. In W. Blum, P.L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 57–68). Springer. https://doi.org/10.1007/978-0-387-29822-1_4

Cramer, K. (2003). Using a translation model for curriculum development and classroom instruction. In R. Lesh & H. Doerr (Eds.), *Beyond Constructivism: Models and Modeling*

Perspectives on Mathematics Problem Solving, Learning, and Teaching (pp. 449–464). Lawrence Erlbaum Associates.

Crompton, H. (2013). A historical overview of mobile learning: Toward learner-centered education. In Z. L. Berge & L. Muilenburg (Eds.), *Handbook of mobile learning* (pp. 3–14). Routledge. https://doi.org/10.4324/9780203118764

Crompton, H., & Burke, D. (2020). Mobile learning and pedagogical opportunities: A configurative systematic review of PreK-12 research using the SAMR framework. *Computers & Educa*tion, 156, 103945. https://doi.org/10.1016/j.compedu.2020.103945

Cropley, A. J. (2001). *Creativity in education & learning: A guide for teachers and educators*. Routledge. https://doi.org/10.4324/9780203826270

Csikszentmihalyi, M. (1988). The dangers of originality: Creativity and the artistic process. In M. M. Gedo (Ed.), *Psychoanalytic Perspectives on Art* (pp. 213–225). Routledge.

Csikszentmihalyi, M. (1999). Implications of a systems perspective. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 325–339). Cambridge University Press.

Csikszentmihalyi, M., & Getzels, J. W. (1971). Discovery-oriented behavior and the originality of creative products: A study with artists. *Journal of Personality and Social Psychology*, *19*(1), 47–52. https://psycnet.apa.org/doi/10.1037/h0031106

Czocher, J. A. (2018). How does validating activity contribute to the modeling process? *Educational Studies in Mathematics*, *99*(2), 137–159. https://doi.org/10.1007/s10649-018-9833-4

Dabbagh, N., & Bannan-Ritland, B. (2005). Online learning: Concepts, strategies, and application. *Educational Technology Research and Development, 55*, 667–669. https://doi.org/10.1007/S11423-007-9071-4

Daher, W. (2010). Building mathematical knowledge in an authentic mobile phone environment. *Australasian Journal of Educational Technology*, *26*(1), 85–104. https://doi.org/10.14742/ajet.1104

Daher, W. M., & Shahbari, J. A. (2015). Pre-service teachers' modelling processes through engagement with model eliciting activities with a technological tool*. International Journal of Science and Mathematics Education*, *13*(S1), 25–46. https://doi.org/10.1007/s10763-013-9464-2

Davis, G. A. (1991). Teaching creativity thinking. In N. Colangelo & G. A. Davis (Eds.), *Handbook of gifted education* (pp. 236–244). Allyn & Bacon.

Davis, G. A., & Rimm, S. B. (1985). *Education of the gifted and talented*. Prentice Hall.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Heath.

Diefes-Dux, H. A., Hjalmarson, M. A., Zawojewski, J. S., & Bowman, K. J. (2006). Quantifying aluminum crystal size part 1: The model-eliciting activity. *Journal of STEM Education: Innovations and Research,* *7*(1&2), 51–63. https://www.jstem.org/jstem/index.php/JSTEM/article/view/1299

Do, N. T., & Thinh, M. P. (2025). Integrating sustainable development goals into high school curricula: Teachers' perceptions and challenges in the Mekong Delta, Vietnam. *International Research in Geographical and Environmental Education.* https://doi.org/10.1080/10382046.2025.2485917

Doerr, H. M., Ärlebäck, J. B., & Misfeldt, M. (2017). Representations of modelling in mathematics education. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical modelling and applications: Crossing and researching boundaries in mathematics education* (pp. 71–81). Springer.

Doerr, H.M. & Pratt, D. (2008). The learning of mathematics and mathematical modeling. In M. K. Heid, & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Syntheses, cases, and perspectives* (pp. 259–285). NCTM.

Dörfler, W. & Blum, W. (1989). Bericht über die Arbeitsgruppe "Auswirkungen auf die Schule" [Report on the working group 'Impact on schools']. In J. Maaß & Schlöglmann, W. (Eds.), *Mathematik als Technologie? Wechselwirkungen zwischen Mathematik, Neuen Technologien, Ausbildung und Weiterbildung* (pp. 174–188). Deutscher Studien Verlag.

Doyle, W. (1998). Classroom organization and management. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 475–504). Routledge.

Drigas, A., & Pappas, M. (2015). A review of mobile learning applications for mathematics. *International Journal of Interactive Mobile Technologies (IJIM)*, *9*(3), 18–23. https://doi.org/10.3991/ijim.v9i3.4420

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: Design research on the understanding of the concept of parameter*. [Doctoral Dissertation, Utrecht University]. Utrecht University Repository. http://dspace.library.uu.nl/handle/1874/886

Drijvers, P., Ball, L., Barzel, B., Heid, M.K., Cao, Y., Maschietto, M. (2016). Uses of technology in lower secondary mathematics education. In P. Drijvers , L. Ball , B. Barzel, M. K. Heid , Y. Cao, M. Maschietto (Eds.), *Uses of Technology in Lower Secondary Mathematics Education. ICME-13 Topical Surveys* (pp. 1–34). Springer. https://doi.org/10.1007/978-3-319-33666-4_1

Dudek, S. Z., & Cote, R. (1994). Problem finding revisited. In Runco, M. A. (ed.), *Problem Finding Problem Solving and Creativity* (pp. 130–150). Ablex.

Duolingo. (n.d.). *Duolingo: Learn a language for free.* Retrieved January 10, 2025, from https://www.duolingo.com/

Durandt, R. (2021). Design principles to consider when student teachers are expected to learn mathematical modelling. *Pythagoras, 42*(1), Article a618. https://doi.org/10.4102/pythagoras.v42i1.618

Durandt, R., Blum, W., & Lindl, A.. (2022). Fostering mathematical modelling competency of South African engineering students: Which influence does the teaching design have?. *Educational Studies in Mathematics*, *109*(2), 361–381. https://doi.org/10.1007/s10649-021-10068-7

Eck, C., Garcke, H., & Knabner, P. (2008). *Mathematische Modellierung* [Mathematical Modelling]. Springer.

edX. (n.d.). *edX: Free online courses by Harvard, MIT, & more.* Retrieved January 10, 2025, from https://www.edx.org/

Edwards, D. B., Sustarsic, M., Chiba, M., & Perriton, S. (2020). Achieving and monitoring education for sustainable development and global citizenship: A systematic review of the literature. *Sustainability, 12*(10), 1383. https://doi.org/10.3390/su12041383

El-Amin, M. F. (2025). *Mathematical modeling for big data analytics*. Morgan Kaufmann.

Engle, C. E. (1997). Not just a method but a way of learning. In D. Boud & G. Felleti (Eds.), *The challenge of problem-based learning* (pp. 17–27). Kogan Page.

Ennis, R. H. (1987). A taxonomy of critical thinking dispositions and abilities. In J. B. Baron & R. J. Sternberg (Eds.), *Teaching thinking skills: Theory and practice* (pp. 9–26). W.H. Freeman and Company.

European Union (2024a). *Sustainable development in the European Union: Monitoring report on progress towards the SDGs in an EU context* (8th vol.). https://doi.org/10.2785/98370

European Union (2024b). *Europe's choice: Political Guidelines for the Next Generation Commission*. https://commission.europa.eu/document/download/e6cd4328-673c-4e7a-8683-f63ffb2cf648_en?filename=Political%20Guidelines%202024-2029_EN.pdf

Fabian, K., Topping, K. J., & Barron, I. G. (2018). Using mobile technologies for mathematics: Effects on student attitudes and achievement. *Educational Technology Research and Development*, *(66)*, 1119–1139. https://doi.org/10.1007/s11423-018-9580-3

Facione, P., Facione, N., & Giancarlo, C. (2000). The disposition toward critical thinking: Its character, measurement, and relationship to critical thinking skill. *Informal Logic, 20*(1). https://doi.org/10.22329/il.v20i1.2254

Feal Filho, W., Salvia, A. L., & Eustachio, J. H. P. P. (2023). An overview of the engagement of higher education institutions in the implementation of the UN Sustainable Development Goals. *Journal of Cleaner Production, 386*, 135694. https://doi.org/10.1016/j.jclepro.2022.135694

Feldhusen, J. F., & Treffinger, D. J. (1980). *Creative thinking and problem solving in gifted education*. Kendal Hunt Publishing Company.

Ferguson, S. (2020). Online mathematics education, the good, the bad, and the general overview. In J. P. Howard & J. F. Beyers (Eds.), *Teaching and Learning Mathematics Online* (pp. 405-418).

Fesakis, G., Karta, P., & Kozas, K. (2018). Designing math trails for enhanced by mobile learning realistic mathematics education in primary education. *International Journal of Engineering Pedagogy (IJEP)*, *8*(2), 49. https://doi.org/10.3991/ijep.v8i2.8131

Fesakis, G., Triantafyllou, S., Tzioufas, N., Koutsomanoli-Filippaki, D., & Volika, S. (2023). A case study for the pedagogical evaluation of the ASYMPTOTE system. In M. Ludwig, S. Barlovits, A. Caldeira, & A. Moura (Eds.), *Research on STEM education in the digital age. Proceedings of the ROSEDA Conference* (pp. 153–160). WTM. https://doi.org/10.37626/GA9783959872522.0.18

Frejd, P. (2014). *Modes of mathematical modelling: An analysis of how modelling is used and interpreted in and out of school settings*. [PhD dissertation, Linköping University Electronic Press]. https://doi.org/10.3384/diss.diva-103689

Frejd, P., & Bergsten, C. (2018). Professional modellers' conceptions of the notion of mathematical modelling: Ideas for education. *ZDM – Mathematics Education*, *50*(1–2), 1-13. https://doi.org/10.1007/s11858-018-0928-2

Frejd, P., & Vos, P. (2023). The spirit of mathematical modeling – a philosophical study on the occasion of 50 years of mathematical modeling education. *Mathematics Enthusiast*, *21*. 269-300. https://doi.org/10.54870/1551-3440.1626.

Fu, X. (2024). A Study on the Application of Task-Driven Teaching Method in High School Mathematics Teaching: Taking "The Determination of Vertical Line and Plane" as an Example. *Journal of Contemporary Educational Research* *8*(8), 1–7. https://doi.org/10.26689/jcer.v8i8.7820

Galbraith, P. L., Goos, M., Renshaw, P., & Geiger, V. (2003). Technology enriched classrooms: Some implications for teaching applications and modelling. In Q.-X. Ye, W. Blum, K. Houston, & Q.-Y. Jiang (Eds.), *Mathematical Modelling in Education and Culture: ICTMA 10* (pp. 111–125). Horwood. https://doi.org/10.1533/9780857099556.3.111

Galbraith, P., & Stillman, G. A. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM – Mathematics Education*, *38*(2), 143–162. https://doi.org/10.1007/BF02655886

Gardner, H. (1993). *Creating minds*. Basic Books.

Geiger, V. (2011) Factors affecting teachers' adoption of innovative practices with technology and mathematical modelling. In G. Kaiser, W. Blum, R. Borromeo Ferri, G.A. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 305–314). Springer. https://doi.org/10.1007/978-94-007-0910-2_31

Geiger, V. (2024). The need for a critical orientation to mathematical modelling in times of disruption. In H.-S. Siller, V. Geiger, & G. Kaiser (Eds.), *Researching Mathematical Modelling Education in Disruptive Times* (pp. 15–34). Springer. https://doi.org/10.1007/978-3-031-53322-8_2

Geiger, V., & Redmond, T. (2013). Designing mathematical modelling tasks in a technology rich secondary school context. In C. Margolinas (Ed.), *Task design in mathematics education: Proceedings of ICMI Study 22* (pp. 119–128). ICMI. http://hal.archives-ouvertes.fr/hal-00834054

Geiger, V., Galbraith, P. L., Renshaw, P., & Goos, M. (2003). Choosing and using technology for secondary mathematical modelling tasks – choosing the right peg for the right hole. In Q.-X. Ye, W. Blum, K. Houston, & Q.-Y. Jian (Eds.), *Mathematical Modelling in Education and Culture* (pp. 126–140). Elsevier. https://doi.org/10.1533/9780857099556.3.126

Geisler, S., & Rach, S. (2023). Students' interest when combining modelling and experimentation – Is it worth the effort? In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, & M. Tabach (Eds.), *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 339–346). PME

Gellert, U., Jablonka, E. & Keitel, C. (2001). Mathematical literacy and common sense in mathematics education. In B. Atweh, H. Forgasz & B. Nebres (Eds.), *Sociocultural research on mathematics education* (pp. 57–73). Lawrence Erlbaum.

Getzels, J. W. (1975). Problem-finding and the inventiveness of solutions. *The Journal of Creative Behavior, 9*(1), 12–18. https://doi.org/10.1002/j.2162-6057.1975.tb00552.x

Gilbert, J. K. (2004). Models and modelling: Routes to more authentic science education. *International Journal of Science and Mathematics Education*, *2*(2), 115–130. https://doi.org/10.1007/s10763-004-3186-4

Gillmor, S., Poggio, J., & Embretson, S. (2015). Effects of reducing the cognitive load of mathematics test items on student performance. *Numeracy*, *8*(1), Article 4. https://doi.org/10.5038/1936-4660.8.1.4

Glaser, E. M. (1941). An experiment in development of critical thinking. *Teachers College Record the Voice of Scholarship in Education, 43*(5), 1–18. https://doi.org/10.1177/016146814204300507

Goffin, S., & Tull, C. (1985). Problem Solving. Encouraging Active Learning. *Young Children*, *40*, 28–32.

Goldin, G. A. (2020). Mathematical representations. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 566–572). Springer.

Gonzalez, F. (2023). Exploring the affordances of place-based education for advancing sustainability education: the role of cognitive, socio-emotional and behavioural learning. *Education Sciences, 13(7*), Article 676. https://doi.org/10.3390/educsci13070676

Gómez-Gómez, M., & García-Lázaro, D. (2023). Awareness and knowledge of the Sustainable Development Goals in teacher training. *Profesorado, Revista de Currículum y Formación del Profesorado, 27*(3), 243–264. https://doi.org/10.30827/profesorado.v27i3.27948

Goos, M., Carreira, S., & Namukasa, I. K. (2023). Mathematics and interdisciplinary stem education: Recent developments and Future Directions. *ZDM – Mathematics Education*, *55*(7), 1199–1217. https://doi.org/10.1007/s11858-023-01533-z

Grant, M. M. (2019). Difficulties in defining mobile learning: Analysis, design characteristics, and implications. *Educational Technology Research and Development, 67*, 361–388. https://doi.org/10.1007/s11423-018-09641-4

Gravemeijer, K., & Doorman, M. (1999). Context Problems in realistic mathematics education: a calculus course as an example. *Educational Studies in Mathematics*, *39*(1/3), 111–129. https://doi.org/10.1023/a:1003749919816

Greca, I. M., & Moreira, M. A. (2000). Mental models, conceptual models, and modelling. *International Journal of Science Education, 22*(1), 1–11. https://doi.org/10.1080/095006900289976

Greefrath G. & Siller H.-S. (2018). Geogebra as a tool in modelling processes. In L. Ball, P. Drijvers P, S. Ladel, H.-S. Siller, M. Tabach, C. Vale (Eds.), *Uses of technology in primary and secondary mathematics education* (pp. 363–374). Springer International. https://doi.org/10.1007/978-3-319-76575-4_21

Greefrath G., Hertleif C. & Siller H.-S. (2018). Mathematical modelling with digital tools: a quantitative study on mathematising with dynamic geometry software. *ZDM – Mathematics Education, 50*(1), 233–244. https://doi.org/10.1007/s11858-018-0924-6

Greefrath, G., Siller, H.-S., & Ludwig, M. (2017). Modelling problems in German grammar school leaving examinations (Abitur)-Theory and practice. In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the 10th Congress of the European Society for Research in Mathematics Education (CERME10)* (pp. 932–939). Dublin City University & ERME. https://hal.science/hal-01933483v1

Greefrath, G., Siller, H.-S., & Weitendorf, J. (2011). Modelling considering the influence of technology. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in Teaching and Learning of Mathematical Modelling* (pp. 315–329). Springer.

Greer, B., & Verschaffel, L. (2007). Modelling competencies - overview. In W. Blum, P. L. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 219–224). Springer.

Groß-Mlynek, L., Graf, T., Harring, M., Gabriel-Busse, K., & Feldhoff, T. (2022). Cognitive Activation in a Close-Up View: Triggers of high cognitive activity in students during group work phases. *Frontiers in Education*, 7. https://doi.org/10.3389/feduc.2022.873340

Guerrero-Ortiz, C., & Camacho-Machín, M. (2022). Characterizing tasks for teaching mathematics in dynamic geometry system and modelling environments. *Mathematics, 10*(8), 1239. https://doi.org/10.3390/math10081239

Guilford, J. P. (1956). The structure of intellect. *Psychological Bulletin*, *53*, 267–293.

Guilford, J. P. (1967a). *The nature of human intelligence*. McGraw-Hill.

Guilford, J. P. (1975). Varieties of creative giftedness, their measurement and development. *Gifted Child Quarterly*, *19*, 107–121.

Guterres, A. (2024). *The Sustainable Development Goals Report 2024.* United Nations. https://unstats.un.org/sdgs/files/report/2024/SG-SDG-Progress-Report-2024-advanced-unedited-version.pdf

Haines, C., & Crouch, R. (2010). Remarks on a modeling cycle and interpreting behaviours. In R. A. Lesh, P. L. Galbraith, W. Blum, & A. Hurford (Eds.), *Modeling student's modeling competencies. ICTMA 13* (pp. 145–154). Springer.

Hallström, J., & Schönborn, K. J. (2019). Models and modelling for authentic STEM education: Reinforcing the argument. *International Journal of STEM Education, 6*, Article 22. https://doi.org/10.1186/s40594-019-0178-z

Hallström, J., & Schönborn, K. J. (2023). Models and modeling in STEM education: Nature, roles, and implementation. In R. J. Tierney, F. Rizvi, & K. Erkican (Eds.), *International Encyclopedia of Education* (4th ed., pp. 112–116). Elsevier. https://doi.org/10.1016/B978-0-12-818630-5.13038-6

Hamilton, E., Lesh, R., Lester, F., & Brilleslyper, M. (2008). Model-eliciting activities (MEAs) as a bridge between engineering education research and mathematics education research. *Advances in Engineering Education, 1*(2), 1–25.

Hamson, M. J. (2003). The place of mathematical modelling in mathematics education. In S. J. Lamon, W. A. Parker, & K. Houston (Eds.), *Mathematical modelling: A way of life. ICTMA11* (pp. 215-226). Horwood Pub.

Handal, B., Campbell, C., Cavanagh, M., & Petocz, P. (2016). Characterising the perceived value of mathematics educational apps in preservice teachers. *Mathematics Education Research Journal*, *28*(1), 199–221.

Hansen, R., & Hana, G. M. (2015). Problem posing from a modelling perspective. In F. F. Singer, N. Ellerton, J. Cai (Eds.), *Mathematical Problem Posing. Research in Mathematics Education* (pp. 35–46). Springer. https://doi.org/10.1007/978-1-4614-6258-3_2

Hansson, S. O. (2020). Technology and mathematics. *Philosophy & Technology, 33*(1), 117–139. https://doi.org/10.1007/s13347-019-00348-9

Harper, D. (n.d.). Etymology of model. Online Etymology Dictionary. Retrieved January 8, 2025, from https://www.etymonline.com/word/model

Hartmann, L.-M., Krawitz, J., & Schukajlow, S. (2023). Posing and Solving Modelling Problems—Extending the Modelling Process from a Problem Posing Perspective. *Journal für Mathematik-Didaktik*, *44*, 533–561. https://doi.org/10.1007/s13138-023-00223-3

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H.-St., Tabach, M., Trgalova, J., & Moreno, L. (2016). *Uses of technology in upper secondary mathematics education. ICME-13 Topical Surveys.* Springer. https://doi.org/10.1007/978-3-319-42611-2

Henning, H., & Keune, M. (2007). Levels of modelling competencies. In W. Blum, P. L. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 225–232). Springer.

Heymann, H. W. (2003). *Why teach mathematics? A focus on general education* (Vol. 33). Springer. https://doi.org/10.1007/978-94-017-3682-4

Hiebert, J., & Wearne, D. (1997). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction*, *14*(3), 251–283. https://doi.org/10.1207/s1532690xci1403_1

Hitchcock, D. (2024). Critical Thinking. In E. N. Zalta & U. Nodelman (Eds*.), The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University.

Hjalmarson, M. A., Holincheck, N., Baker, C. K., & Galanti, T. M. (2020). Learning models and modelling across the STEM disciplines. In C. C. Johnson, M. J. Mohr-Schroeder, T. J. Moore, & L. D. English (Eds.), *Handbook of Research on STEM Education* (pp. 223–233). Taylor & Francis.

Hjalmarson, M., Diefes-Dux, H. A., Bowman, K., & Zawojewski, J. S. (2006). Quantifying aluminum crystal size part 2: The model development sequence. *Journal of STEM Education: Innovations and Research, 7*(1/2), 64–73.

Hoffmann, K.-H., & Witterstein, G. (2014). *Mathematische Modellbildung: Grundprinzipien in Natur- und Ingenieurwissenschaften* [Mathematical modeling: Basic principles in natural sciences and engineering]. Springer.

Honebein, P. C., Duffy, T. M., & Fishman, B. J. (1993). Constructivism and the design of learning environments: Context and authentic activities for learning. In T.M. Duffy, J.

Lowyck & D. H. Jonassen (Eds.), *Designing environments for constructivist learning* (pp. 87–108). Springer-Verlag.

Hopkinson, P. & James, P. (2010). Practical pedagogy for embedding ESD in science, technology, engineering and mathematics curricula. *International Journal of Sustainability in Higher Education*, *11*(4), 365–379. https://doi.org/10.1108/14676371011077586

Hsu, H., & Yao, C.-Y. (2023). A review of the mathematical tasks framework and levels of cognitive demand. In J. Cai, G. J. Stylianides, P. A. Kenney (Eds.), *Research Studies on Learning and Teaching of Mathematics. Research in Mathematics Education* (pp. 219–252). Springer. https://doi.org/10.1007/978-3-031-35459-5_10

Huan, Y., Yu, Y., Liang, T., & Burgman, M. (2021). A method for assessing the impacts of an international agreement on regional progress towards Sustainable Development Goals. *Science of the Total Environment, 785*, 142302. https://doi.org/10.1016/j.scitotenv.2021.147336

Huang, Y., Chen, Y., Liu, S., & Zhang, Y. (2025). Mamo: Evaluating mathematical modeling capability of large language models. *arXiv preprint arXiv*, 2404.12345. https://arxiv.org/abs/2404.12345

Hung, W., Jonassen, D. H., & Liu, R. (2008). Problem-Based Learning. In J. M. Spector, J. G. van Merrienboer, M. D., Merrill, & M. Driscoll (Eds*.), Handbook of Research on Educational Communications and Technology* (3 ed., pp. 485–506). Erlbaum.

Ibáñez, M.-B., & Delgado-Kloos, C. (2018). Augmented reality for STEM learning: A systematic review. *Computers & Education*, *123*, 109–123. https://doi.org/10.1016/j.compedu.2018.05.002

ILIAS. (n.d.). *ILIAS: Open source e-learning*. Retrieved January 10, 2025, from https://www.ilias.de/en/

Jablonka, E. (1996). *Meta-Analyse von Zugängen zur mathematischen Modellbildung und Konsequenzen für den Unterricht* [Doctoral dissertation, Technische Universität Berlin]. Transparent Verlag.

Jablonka, E., & Gellert, U. (2007). Mathematisation – dematematisation. In U. Gellert, & E. Jablonka (Eds.), *Mathematisation and demathematisation: Social, philosophical and educational ramifications* (pp. 1–18). Sense Publishers.

Jablonski, S. (2024). Challenges in geometric modelling—A comparison of students' mathematization with real objects, photos, and 3D models. *Eurasia Journal of Mathematics, Science and Technology Education, 20*(3), em2414. https://doi.org/10.29333/ejmste/14321

Jablonski, S., Barlovits, S., & Ludwig, M. (2023). How digital tools support the validation of outdoor modelling results. *Frontiers in Education*, *8*, Article 1145588. https://doi.org/10.3389/feduc.2023.1145588

Jablonski, S., Ehemann, J., Oehler, D.-X. K., Ludwig, M. (in press). One step at a time? – An empirical study on the use of graphical representations for solving Fermi problems. In *Proceedings of the 14th Congress of the European Society for Research in Mathematics Education (CERME14)*. Free University of Bozen-Bolzano & ERME.

Jablonski, S., Barlovits, S., Ludwig, M., Ärlebäck, J. B., Carreira, S. Fesakis, G., Fock, A., Jacinto, H., Julsgård, S., Moustaka, M. A., Lázaro del Pozo, C., Reyes Delgado, J. A., & Siller, H.-S. (submitted). Teachers' perspectives on mathematical modelling: A European view on characteristics, benefits and challenges. In *International Perspectives on the Teaching and Learning of Mathematical Modelling*. Springer.

Jacobs, Mary & Dominowski, Roger. (2013). Learning to solve insight problems*. Bulletin of the Psychonomic Society*, *17*, 171–174. https://doi.org/10.3758/BF03333702.

Jeppsson, U., & Haglund, L. (2003). *Modeller och metaforer: En introduktion till vetenskapsteori*. [Models and Metaphors: An Introduction to the Theory of Science]. Studentlitteratur.

Jones, N. A., Ross, H., Lynam, T., Perez, P., & Leitch, A. (2011). Mental models: An interdisciplinary synthesis of theory and methods. *Ecology and Society*, *16*(1), 46.

Julie, C., & Mudaly, V. (2007). Mathematical modelling of social issues in school mathematics in South Africa. . In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education. The 14th ICMI Study* (pp. 503–510). Springer. https://doi.org/10.1007/978-0-387-29822-1_58

Just, J, Siller, H.-S. & Vorhölter, K. (2023). Bildung für Nachhaltige Entwicklung im Mathematikunterricht am Beispiel des Themas Klima [Education for Sustainable Development in Mathematics Teaching Using the Example of Climate]. *MNU Journal*, *6*, 456–463.

Kaiser, G. (2020). Mathematical Modelling and Applications in Education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 553–561). Springer. https://doi.org/10.1007/978-3-030-15789-0_101

Kaiser, G., & Sriraman, B. (2006). A global survey of International Perspectives on Modelling in Mathematics Education. *ZDM – Mathematics Education*, *38*(3), 302–310. https://doi.org/10.1007/bf02652813

Kaiser, G., Blomhøj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modelling*. ZDM – Mathematics Education*, *38*(2), 82–85.

Kaiser, G., Sriraman, B., Blomhøj, M., & García, F. J. (2007). Report from the working group modelling and applications - Differentiating perspectives and delineating commonalties. In D. Pitta-Pantazi & G. Philippou (Eds*.), European Research in Mathematics Education V (Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education)* (pp. 2035–2041). University of Cyprus.

Kaur, B., & Chin, S. L. (2022). Nature of mathematics tasks and what teachers do. *Current Opinion in Behavioral Sciences*, *46*, 101169. https://doi.org/10.1016/j.cobeha.2022.101169

Kertil, M., & Gurel, C. (2016). Mathematical Modeling: A Bridge to STEM Education. *International Journal of Education in mathematics, science and Technology*, *4*(1), 44–55.

Kiper, H., Meyer, H., Mischke, W., & Wester, F. (2004). *Qualitätsentwicklung in Unterricht und Schule. Das Oldenburger Konzept* [Quality Development in Teaching and School: The Oldenburg Concept]. diz.

Kitchener, K. S. & Fischer, K. W. (1990). A skill approach to the development of reflective thinking. In D. Kuhn (Ed.), *Contributions to human development: Vol. 21. Developmental perspectives on teaching and learning thinking skills* (pp. 48–62). Karger.

Klavir, R., & Hershkovitz, S. (2008). Teaching and evaluating 'open-ended' problems. *International Journal for Mathematics Teaching and Learning*, *20*. Retrieved January 7, 2025, from https://www.cimt.org.uk/journal/klavir.pdf

Klieme E., Schümer G., & Knoll S. (2001) Mathematikunterricht in der Sekundarstufe I: "Aufgabenkultur" und Unterrichtsgestaltung im internationalen Vergleich [Mathematics instruction at lower secondary level: 'Task culture' and quality of instruction in international comparison]. In E. Klieme, & J. Baumert (Eds.), *TIMSS – Impulse für Schule und Unterricht: Forschungsbefunde, Reforminitiativen, Praxisberichte und Video-Dokumente* (pp. 43–57). BMBF.

Klock, H., & Siller, H.-S. (2020). A Time-Based Measurement of the Intensity of Difficulties in the Modelling Process. In G. A. Stillman, G. Kaiser, & C. E. Lampen (Eds.), *Mathematical*

Modelling Education and Sense-making. International perspectives on the teaching and learning of mathematical modelling (pp. 163–173). Springer. https://doi.org/10.1007/978-3-030-37673-4_15

KMK (2024). *Empfehlung der Kultusministerkonferenz zur Bildung für nachhaltige Entwicklung in der Schule (Beschluss der Kultusministerkonferenz vom 13.06.2024)* [Recommendation of the Standing Conference of the Ministers of Education and Cultural Affairs on Education for Sustainable Development in Schools]. https://www.kmk.org/fileadmin/veroeffentlichungen_beschluesse/2024/2024_06_13-BNE-Empfehlung.pdf

Krutikhina, M. V., Vlasova, V. K., Galushkin, A. A., & Pavlushin, A. A. (2018). Teaching of mathematical modeling elements in the mathematics course of the secondary school. *Eurasia Journal of Mathematics, Science and Technology Education, 14*(4), 1305–1315. https://doi.org/10.29333/ejmste/83561

Kumar, P., Ahmed, F., Singh, R. K., & Sinha, P. (2018). Determination of hierarchical relationships among sustainable development goals using interpretive structural modeling. *Environment, Development and Sustainability, 20*(5), 2119–2137. https://doi.org/10.1007/s10668-017-9981-1

Kunter, M., & Voss, T. (2013). The model of Instructional Quality in COACTIV: a multicriteria analysis. In M. Kunter, J. Baumert, W. Blum, U. Klusmann, S. Krauss, & M. Neubrand (Eds.), *Cognitive Activation in the Mathematics Classroom and Professional Competence of Teachers. Results from the COACTIV Project* (pp. 97–124). Springer. https://doi.org/10.1007/978-1-4614-5149-5_6

Lafuente-Lechuga, M., Cifuentes-Faura, J. & Faura-Martínez, Ú. (2020). Mathematics Applied to the Economy and Sustainable Development Goals: A Necessary Relationship of Dependence. *Education Sciences*, *10*(11), 339. https://doi.org/10.3390/educsci10110339

Larkin, K., & Calder, N. (2016). Mathematics education and mobile technologies. *Mathematics Education Research Journal*, *28*(1), 1–7. https://doi.org/10.1007/s13394-015-0167-6

Läufer, T., Stäter, R. S., & Ludwig, M. (2024). Fostering Algorithmic Thinking in Mathematics Education with AR. In F. Dilling, & I. Witzke (Eds.), *Augmented and Virtual Reality in Mathematics Education. International Symposium on Augmented and Virtual Reality 2022* (pp. 121–131). Springer. https://doi.org/10.1007/978-3-658-45271-1_7

Le, H. C., Nguyen, V. H., & Nguyen, T. L. (2023). Integrated STEM approaches and associated outcomes of K-12 student learning: a systematic review. *Education Sciences*, *13*(3), 297.

Ledezma, C., Breda, A., & Font, V. (2024). Prospective teachers' reflections on the inclusion of mathematical modelling during the transition period between the face-to-face and virtual teaching contexts. *International Journal of Science and Mathematics Education*, *22*(5), 1057–1081. https://doi.org/10.1007/s10763-023-10412-8

Lee, Z. (2022). Mathematical model and its classifications. *Research & Reviews: Journal of Statistics and Mathematical Sciences, 8*(4), Article 003. https://doi.org/10.4172/JStatsMathSci.8.4.003

Leong, K. E., & Tan, J. Y. (2020). Exploring secondary students' modelling competencies. *The Mathematics Enthusiast, 17*(1), 85–107. https://doi.org/10.54870/1551-3440.1481

Lesh R., Post, T. & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 33–40). Lawrence Erlbaum Associates.

Lesh, R. (1979). Mathematical learning disabilities: Considerations for identification, diagnosis and remediation. In R. Lesh, D. Meierkiewicz, & M. G. Kantowski (Eds.), *Applied mathematical problem solving*. ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Lesh, R. A., & Doerr, H. M. (2003a). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3–33). Routledge.

Lesh, R. A., & Doerr, H. M. (Eds.) (2003b). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Routledge. https://doi.org/10.4324/9781410607713

Lester, F. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *ZDM – Mathematics Education*, *37*(6), 457–467. https://doi.org/10.1007/BF02655854

Li, H.-C. & Tsai, T.-L. (2022). Education for sustainable development in mathematics education: what could it look like? *International Journal of Mathematical Education in Science and Technology*, *53*(9), 2532–2542. https://doi.org/10.1080/0020739X.2021.1941361

Liang, S. (2016). Teaching the concept of limit by using conceptual conflict strategy and Desmos graphing calculator. *International Journal of Research in Education and Science (IJRES)*, *2*(1), 35–48.

Lillis, M. (2024). The (A) of STE(A)M. The contribution of Art to STE(A)M Education. *Hellenic Journal of STEM Education*, *3*(2), 21–27.

Lingefjärd, T. (2006). Faces of mathematical modeling. *ZDM – Mathematics Education*, *38*(2), 96–112.

Lipman, M. (1991). *Thinking in Education*. Cambridge University Press.

Lompscher, J. (1985). *Persönlichkeitsentwicklung in der Lerntätigkeit*. Volk u. Wissen.

Lu, X., & Kaiser, G. (2022). Creativity in students' modelling competencies: conceptualisation and measurement. *Educational Studies in Mathematics*, *109*, 287–311.

Luczak, R., & Erwin, R. (2023). Mathematical modeling: A study of multidisciplinary benefits in the math classroom. *Teaching Mathematics and Its Applications, 42*(4), 325–342. https://doi.org/10.1093/teamat/hrac02

Ludwig, M., & Jesberg, J. (2015). Using Mobile Technology to Provide Outdoor Modelling Tasks - The MathCityMap-Project. *Procedia - Social and Behavioral Sciences*, *191*, 2776–2781. https://doi.org/10.1016/j.sbspro.2015.04.517

Maaß, K. (2006). What are modelling competencies? *ZDM – Mathematics Education*, *38*(2), 113–142. https://doi.org/10.1007/BF02655885

Maaß, K. (2010). Classification scheme for modelling tasks. *Journal für Mathematik-Didaktik, 31*(2), 285–311. https://doi.org/10.1007/s13138-010-0010-2

Maass, K., Geiger, V., Ariza, M. R., & Goos, M. (2019). The role of mathematics in interdisciplinary STEM education. *ZDM – Mathematics Education*, *51*(6), 869–884. https://doi.org/10.1007/s11858-019-01100-5

Mansfield, R. S., Busse, T. V., Krepelka, E. G. (1978). The effectiveness of creative training. *Review of Educational Research*, *48*(4), 517–536. https://www.jstor.org/stable/1170047

Martinsen, Ø. (1995). Cognitive styles and experience in solving insight problems: Replication and extension. *Creativity Research Journal*, *8*(3), 291–298. https://doi.org/10.1207/s15326934crj0803_8

Mazumdar, A., Burkotová, J., Krátký, T., Chugh, T., & Miettinen, K. (2024). Handling simulation failures of a computationally expensive multiobjective optimization problem in pump design. *Engineering Applications of Artificial Intelligence, 136* (Part A), Article 108897. https://doi.org/10.1016/j.engappai.2024.108897

Merriam-Webster. (n.d.). *Model*. In Merriam-Webster.com dictionary. Retrieved February 17, 2025, from https://www.merriam-webster.com/dictionary/model

Milicic, G., Borkulo, S. P. van, Medova, J., Wetzel, S., & Ludwig, M. (2021). Design and development of a learning environment for computational thinking: The Erasmus+ project. In *EDULEARN21 Proceedings* (pp. 7376–7383). IATED. https://doi.org/10.21125/edulearn.2021.1495

Molina-Toro, J. F., Rendón-Mesa, P. A., & Villa-Ochoa, J. A. (2019). Research trends in digital technologies and modeling in mathematics education. *Eurasia Journal of Mathematics, Science and Technology Education*, *15*(8), em1736. https://doi.org/10.29333/ejmste/108438

Molyneux-Hodgson, S., Rojano, T., Sutherland, R., & Ursini, S. (1999). Mathematical modelling: The interaction of culture and practice. *Educational Studies in Mathematics*, *39*(1–3), 167–183. https://doi.org/10.1023/A:1003758310286

Monzó-Martínez, A., Ortiz-Cermeño, E., & Martínez-Agut, M. P. (2024). Global citizenship education and its role in sustainability at the university Level. *Education Sciences, 14*(8), 847. https://doi.org/10.3390/educsci14080847

Mullis, I. V., & Martin, M. O. (2017). TIMSS 2019 assessment frameworks. *International Association for the Evaluation of Educational Achievement*. https://timssandpirls.bc.edu/timss2019/frameworks/index.html

Murthy, D. N. P., Page, N. W., & Rodin, E. Y. (1990). *Mathematical modelling: A tool for problem solving in engineering, physical, biological, and social sciences* (1st ed.). Pergamon Press.

Nickerson, R. S. (1987). Why teach thinking? In J. B. Baron, & R.J. Sternberg (Eds.). *Teaching thinking skills: Theory and practice* (pp. 27–37). W. H. Freeman and Company.

Ningsih, S. Y. & Juandi, D. (2019). Achievement of ESD (Educational for Sustainable Development) through mathematics learning. *Journal of Physics: Conference Series*, 1157, 42056. https://doi.org/10.1088/1742-6596/1157/4/042056

Niss, M. (1989). Aims and scope of applications and modelling in mathematics curricula. In W. Blum, J. S. Berry, R. Biehler, I. Huntley, G. Kaiser-Messmer, & L. Profke (Eds.), *Applications and modelling in learning and teaching mathematics* (pp. 22–31). Ellis Horwood.

Niss, M. (1996). Goals of mathematics teaching. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 11–47). Kluwer Academic Publishers.

Niss, M. (2010). Modeling a crucial aspect of students' mathematical modeling. In R. Lesh, P. Galbraith, C. R. Haines, & A. Hurford (Eds*.), Modelling students' mathematical competencies* (pp. 43–59). Springer.

Niss, M. A. (2003). Mathematical competencies and the learning of mathematics: the Danish KOM project. In A. Gagatsis, & S. Papastavridis (Eds.), *3rd Mediterranean Conference on Mathematical Education* -Athens, Hellas 3-4-5 January 2003 (pp. 116–124). Hellenic Mathematical Society.

Niss, M., & Blum, W. (2020). *The teaching and learning of mathematical modelling*. Routledge. https://doi.org/10.4324/9781315189314

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 3–32). Springer. https://doi.org/10.1007/978-0-387-29822-1_1

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics*, *102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jensen, T. H. (Eds.). (2002). *Kompetencer og matematiklæring* [Competencies and mathematics learning]. Undervisningsministeriet.

Ogborn, J. (1994). Overview: the nature of modelling. In J. Mellar, R. Bliss, R. Booha, J. Ogborn, & C. Tompsett (Eds.), *Learning with artificial worlds: Computer based modelling in the curriculum* (pp. 11–15). Falmer press.

Ortiz-Revilla, J., Adúriz-Bravo, A., & Greca, I. M. (2020). A framework for epistemological discussion on integrated STEM education. *Science & Education, 29*(4), 857–880. https://doi.org/10.1007/s11191-020-00131-9

Oxford English Dictionary. (2024, December). *Model* (n. & adj.). In the Oxford English Dictionary. https://doi.org/10.1093/OED/3984201854

Paget, C. (2004). Inquiry and problem-based learning in science classrooms. *Premier's Teacher Scholarship*, 2003-2004, 120–126.

Palmgren, E., & Rasa, T. (2024). Modelling roles of mathematics in physics: Perspectives for physics education. *Science & Education, 33*(2), 365–382. https://doi.org/10.1007/s11191-022-00393-5

Parnes, S. J., & Brunelle, E. A. (1967). The literature of creativity (Part 1). *Journal of Creative Behavior*, *1*(1), 52–109. https://doi.org/10.1002/j.2162-6057.1967.tb00010.x

Paul, R. (1990). *Critical thinking: What every person needs to survive in a rapidly changing world.* Santa Rosa.

Paul, R. W., Elder, L., & Bartell, T. (1997). *California teacher preparation for Instruction in critical thinking: Research findings and policy recommendations.* California Commission of Teacher Credentialing.

Photomath. (n.d.). *Photomath: Your math problem solver*. Retrieved January 10, 2025, from https://photomath.com/

Pierce, R. (2005). Algebraic insight underpins the use of CAS for modelling. Mathematics Enthusiast, *2*(2), 107–117. https://doi.org/10.54870/1551-3440.1027

Pinto, C. M. A., Zeidan, D., Cortés López, J. C., & Machado, J. A. T. (2023). Editorial of the special issue on modelling, analysis, and applications. *Mathematical Methods in The Applied Sciences*. https://doi.org/10.1002/mma.9873

Pollak, H. O. (1979). The interaction between mathematics and other school subjects. In UNESCO (Ed.), *New trends in mathematics teaching IV* (pp. 232–248). UNESCO.

Praetorius, A., Klieme, E., Herbert, B., & Pinger, P. (2018). Generic dimensions of teaching quality: the German framework of Three Basic Dimensions. *ZDM – Mathematics Education*, *50*(3), 407–426. https://doi.org/10.1007/s11858-018-0918-4

Radmehr, F. (2023). Toward a theoretical framework for task design in mathematics education. *Journal on Mathematics Education*, *14*(2), 189–204. https://doi.org/10.22342/jme.v14i2.pp189-204

Raissi, M., Perdikaris, P., & Karniadakis, G. E. (2019). Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, *378*, 686–707. https://doi.org/10.1016/j.jcp.2018.10.045

Resnick, L. B. (1987). *Education and Learning to Think*. London Falmer. National Academy Press.

Rezat, S., & Strässer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: Artifacts as fundamental constituents of the didactical situation. *ZDM – Mathematics Education*, *44*(5), 641–651. https://doi.org/10.1007/s11858-012-0448-4

Ristic, B., Gilliam, C., Byrne, M., & Benavoli, A. (2019). A tutorial on uncertainty modeling for machine reasoning. *Information Fusion*, *55*, 30–44. https://doi.org/10.1016/j.inffus.2019.08.001

Rose, L. H., & Lin, H. T. (1984). A meta-analysis of long-term creativity training programs, *Journal of Creative Behaviour*, *18*(1), 11–22.

Runco, M. A. (1997). *Creativity research handbook*, (Vols 1-3). Hampton Press.

Runco, M. A. (2000). Creativity: Research on the process of creativity. In A. E. Kazdin (Ed.), *Encyclopedia of psychology* (Vol. 2, pp 342–346). American Psychological Association.

Runco, M. A., Chand, I. (1995). Cognition and creativity. *Educational Psychology Review*, *7*(3), 243–267. https://doi.org/10.1007/BF02213373

Runco, M. A., Okuda, S. M. (1988). Problem-discovery, divergent thinking, and the creative process. *Journal of Youth and Adolescence*. *17*(3), 211–220. https://doi.org/10.1007/BF01538162

Sangwin, C. (2007). A brief review of GeoGebra: Dynamic mathematics. *MSOR Connections*, *7*(2), 36.

Savery, J. R., & Duffy, T. M. (1995). Problem-based learning: An instructional model and its constructivist framework. *Educational Technology*, *35*(5), 31–37.

Schaap, S., Vos, P. & Goedhart, M. (2011). Students overcoming blockages while building a mathematical model: exploring a framework. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G.A. Stillman (Eds.) *Trends in teaching and learning of mathematical modelling*. Springer Netherlands (pp. 137–146). https://doi.org/10.1007/978-94-007-0910-2_15

Schafersman, S. D. (1991). An introduction to critical thinking. URL (last checked 28 March 2011). Available at https://smartcollegeplanning.org/wp-content/uploads/2010/03/Critical-Thinking.pdf

Scheja, B., & Rott, B. (2024). Mathematical tasks: A review of classification systems. *Mathematics Education Research Journal*. Advance online publication. https://doi.org/10.1007/s13394-024-00506-z

Schlange, L.E., Jüttner, U. (1997), Helping managers to identify the key strategic issues. *Long Range Planning*, *30*(5), 777–786. https://doi.org/10.1016/S0024-6301(97)00054-X

Schukajlow, S., Kolter, J. & Blum, W. (2015). Scaffolding mathematical modelling with a solution plan. *ZDM – Mathematics Education*, *47*(7), 1241–1254. https://doi.org/10.1007/s11858-015-0707-2

Schukajlow, S., Krawitz, J., Kanefke, J., Blum, W., & Rakoczy, K. (2023). Open modelling problems: cognitive barriers and instructional prompts. *Educational Studies in Mathematics*, *114*(3), 417–438. https://doi.org/10.1007/s10649-023-10265-6

Shute, V. J., Sun, C., & Asbell-Clarke, J. (2017). Demystifying computational thinking. *Educational Research Review*, *22*, 142–158. https://doi.org/10.1016/j.edurev.2017.09.003

Sefertzi, E. (2013). *Creativity. Report produced for the EC funded project INNOREGIO: Dissemination of innovation and knowledge management techniques, January 2000*. Retrieved from http://www.urenio.org/tools/en/creativity.pdf

Serpe, A. (2025). Applied mathematical modelling in the physics problem-solving classroom. In Y. D. Sergeyev, D. E. Kvasov, & M. J. Piotrowska (Eds.), *Numerical computations: Theory and algorithms* (pp. 193–206). Springer. *Lecture Notes in Computer Science* (Vol. 14478). https://doi.org/10.1007/978-3-031-81247-7_15

Siller, H-S, Geiger, V., & Kaiser, G. (Eds) (2024b). *Researching Mathematical Modelling Education in Disruptive Times.* Springer. https://doi.org/10.1007/978-3-031-53322-8

Siller, H.-S., & Greefrath, G. (2010). Mathematical modelling in class regarding to technology. In V. Durand-Guerrier, S. Soury-Lavergne, F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 2136–2145). ERME.

Siller, H.-S., Bruder, R., Hascher, T., Linnemann, T., Steinfeld, J. et al. (2015). Competency level modelling for school leaving examination. In *CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education* (p. 2716-2723), ERME, Prague, Czech Republic.

Siller, H.-S., Geiger, V., & Greefrath, G. (2023a). The role of digital resources in mathematical modelling in extending mathematical capability. In B. Pepin, G. Gueudet, J. Choppin (Eds.), *Handbook of Digital Resources in Mathematics Education. Springer International Handbooks of Education* (pp. 495–518). Springer. https://doi.org/10.1007/978-3-030-95060-6_18-1

Siller, H.-S., Nitzan-Tamar, O., & Kohen, Z. (2023b). Scaffolding practices for modelling instruction in STEM-related contexts: Insights from expert and novice teachers. *ZDM – Mathematics Education, 55*(6), 1483–1498. https://doi.org/10.1007/s11858-023-01529-9

Siller, H.-S., Vorhölter, K., & Just, J. (2024a). Problem posing as a way of promoting individual mathematical thinking in STEM contexts – The case of climate change. *International Journal of Science and Mathematics Education*, *22*(1), Article 10518. https://doi.org/10.1007/s10763-024-10518-7

Siller, H.-S., Günster, S. M., & Geiger, V. (2024b). Mathematics as a central focus in STEM – Theoretical and practical insights from a special study program within pre-service (prospective) teacher education. In Y. Li, Z. Zeng & N. Song (Eds.), *Advances in STEM education. Disciplinary and interdisciplinary education in STEM* (S. 317–343). Springer. https://doi.org/10.1007/978-3-031-52924-5_15

Siller, H.-S., Vorhölter, K., Just, J., Orschulik, A. & Zieriacks, C. (2025b). Empirical Differentiation of Student Competencies in ESD. In *ICMI Study 27*, Quezon City, Philippinen.

Siller, H.-S., Vorhölter, K., Oldenburg, R., Schneider, K., Wagener, M. & Warmeling, A. (2025). Mathematik [Mathematics]. In Engagement Global gGmbH (Hrsg.), *KMK/BMZ Orientierungsrahmen Globale Entwicklung – Bildung für nachhaltige Entwicklung in der gymnasialen Oberstufe* (S. 591–618). Westermann.

Siller, H.-S. & Vorhölter, K. (2025). Mathematical modelling as part of a general education in mathematics classrooms - challenge for teachers, responsibility for researchers. In *Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education (CERME14)*, Free University of Bozen-Bolzano; ERME, Feb 2025, Bozen-Bolzano, Italy. ⟨hal-05201510⟩

Silver, E. A. (1994). On mathematical problem posing. *For the learning of mathematics*, *14*(1), 19–28.

Simon, H. A. (1973). The structure of ill structured problems. *Artificial intelligence*, *4*(3–4), 181–201. https://doi.org/10.1016/0004-3702(73)90011-8

Skov Hansen, N., Holm, C., & Troels-Smith, K. (1999). Modelkompetencer [Model Competencies]. *NOMAD Nordic Studies in Mathematics Education*, *3-4*(7), 7-33. https://doi.org/10.7146/nomad.v7i3-4.146692

Skovsmose, O. (2023). *Critical Mathematics Education*. Springer International Publishing. https://doi.org/10.1007/978-3-031-26242-5

Sokolowski, J. A., & Banks, C. M. (2012). *Handbook of real-world applications in modeling and simulation*. Wiley. https://doi.org/10.1002/9781118241042

Sriraman, B., & Kaiser, G. (2006). Theory usage and theoretical trends in Europe: A survey and preliminary analysis of CERME4 research reports. *ZDM – Mathematics Education*, *38*(1), 22–51. https://doi.org/10.1007/BF02655904

Sriraman, B., Kaiser, G., & Blomhøj, M. (2006). A brief survey of the state of mathematical modeling around the world. *ZDM – Mathematics Education*, *38*(3), 212–213. https://doi.org/10.1007/BF02652805

Starko, A. J. (2005). *Creativity in the classroom: Schools of curious delight* (3rd ed.). Mahwah, NJ: L. Erlbaum Associates.

Steffensen, L. (2020). climate change and students' critical competencies: a norwegian study. In J. Anderson, J., & Y. Li (Eds.), *Integrated Approaches to STEM Education. Advances in STEM Education* (pp. 271–293). Springer. https://doi.org/10.1007/978-3-030-52229-2_15

Sternberg, R. J., Lubart, T. I. (1996). Investing in creativity. *American Psychologist*, *51*(7), 677–688. https://doi.org/10.1037/0003-066X.51.7.677

Sternberg, R. J., Lubart, T. I. (1991). Creating creative minds. *Phi Delta Kappan*, *72*(8), 608–614.

Stillman, G. A. (2019). State of the Art on Modelling in Mathematics Education—Lines of Inquiry. In G. A. Stillman & J. P. Brown (Eds.), *Lines of Inquiry in Mathematical Modelling Research in Education* (ICME-13 Monographs, pp. 1–20).

https://doi.org/10.1007/978-3-030-14931-4_1

Stillman, G. A., Ikeda, T., Schukajlow, S., de Loiola Araújo, J., & Ärlebäck, J. B. (2023). Survey of interdisciplinary aspects of the teaching and learning of mathematical modelling in mathematics education. In G. Greefrath, S. Carreira, & G. A. Stillman (Eds.), *Advancing and consolidating mathematical modelling: Research from ICME-14* (pp. 21–41). Springer. https://doi.org/10.1007/978-3-031-27115-1_2

Stillman, G., & Brown, J. P. (2012). Empirical evidence for Niss' implemented anticipation in mathematising realistic situations. In J. Dindyal, L. P. Cheng, & S. F. Ng (Ed.). *Mathematics education: Expanding horizons*. (pp. 682 – 689). Mathematics Education Research Group of Australasia.

Stillman, G., & Brown, J.P. (2014). Evidence of implemented anticipation in mathematising by beginning modellers. *Mathematics Education Research Journal*, *26*(4), 763–789 (2014). https://doi.org/10.1007/s13394-014-0119-6

Stillman, G., Brown, J., & Galbraith, P. (2009). Identifying Challenges within Transition Phases of Mathematical Modeling Activities at Year 9. In R. Lesh, P. Galbraith, C. Haines, & A. Hurford (Eds.). *Modeling Students' Mathematical Modeling Competencies*. (pp. 385–398). Springer. https://doi.org/10.1007/978-1-4419-0561-1_33

Storey, V. C., & Baskerville, R. L. (2024). Digitalization of the natural sciences: Design science research and computational science. *Decision Support Systems*, *189*, Article 114368. https://doi.org/10.1016/j.dss.2024.114368

Suárez, Á., Specht, M., Prinsen, F., Kalz, M., & Ternier, S. (2018). A review of the types of mobile activities in mobile inquiry-based learning*. Computers & Education*, *118*, 38–55. https://doi.org/10.1016/j.compedu.2017.11.004

Subotnik, R. F. (1988). Factors from the structure of intellect model associated with gifted adolescents' problem finding in science Research with Westinghouse science talent search winners. *Journal of Creative Behavior*, *22(1)*, 42–54.

Sun, Y., Ren, J., Shen, G., & Chen, X. (2024). Research on model optimization based on ISM and DEMATEL. In *Proceedings of the 2024 13th IEEE International Conference on Communication Systems and Network Technologies (CSNT 2024)*. IEEE.

Swetz, F. J. (1989). An historical example of mathematical modelling: The trajectory of a cannonball. *International Journal of Mathematical Education in Science and Technology*, *20*(5), 731–741. https://doi.org/10.1080/0020739890200511

Swetz, F., & Hartzler, J.,S. (1991). *Mathematical modeling in the secondary school curriculum*. National Council of Teachers of Mathematics.

Taylor, C. W. (1972). Can organisations be creative, too?. In C.W. Taylor (Ed.), *Climates for Creativity*, Pergamon Press, pp. 1-15.

Teig, N., Scherer, R., & Nilsen, T. (2019). I know I can, but do I have the time? the role of teachers' self-efficacy and perceived time constraints in implementing cognitive-activation strategies in science. *Frontiers in Psychology*, *10*. https://doi.org/10.3389/fpsyg.2019.01697

Torrance, E. P. (1966). *Torrance Tests of Creative Thinking: Norms-Technical Manual: Verbal Tests, Forms A and B: Figural Tests, Forms A and B (Research Edition)*. Personnel Press.

Torrance, E. P., (1990). *Torrance Tests of Creative Thinking: Manual for scoring and interpreting results*. Verbal Forms A &B. Scholastic Testing Service, Inc.

Torres, D. I. R. (2021). Contribution of higher education to the Sustainable Development Goals from teaching. *Revista Española de Educación Comparada, 37*, 89–110. https://dialnet.unirioja.es/servlet/articulo?codigo=7712086

Traxler, J. (2005). Defining mobile learning. In P. Isaias, c. Borg, P. Kommers & P. Bonnanno (Eds.), In *Proceedings of IADIS International Conference on Mobile Learning* (pp. 261-266). Qawra, Maldives: IADIS

Uitto, A. & Saloranta, S. (2017). Subject teachers as educators for sustainability: a survey study. *Education Sciences*, *7*(1), 8. https://doi.org/10.3390/educsci7010008

UNESCO (2015). *Rethinking education: towards a global common good?* UNESCO Publishing. https://doi.org/10.54675/MDZL5552

UNESCO (2017). *Education for sustainable development goals: Learning objectives. Division for Inclusion, Peace and Sustainable Development*, Education Sector. https://doi.org/10.54675/CGBA9153

United Nations (2015). *Transforming our world: The 2030 Agenda for Sustainable Development. resolution / adopted by the General Assembly*, A/70/L.1. https://sdgs.un.org/sites/default/files/publications/21252030%20Agenda%20for%20Sustainable%20Development%20web.pdf

Upmeier zu Belzen, A., Krüger, D., & van Driel, J. (Eds.). (2019). *Towards a competence-based view on models and modeling in science education.* Springer. https://doi.org/10.1007/978-3-030-30255-9

Van Gelder, T. J. (2001). How to improve critical thinking using educational technology. In G. Kennedy, M. Keppell, C. McNaught, & T. Petrovic (Eds.), *Meeting at the Crossroads. Proceedings of the 18th Annual Conference of the Australasian Society for Computers in Learning in Tertiary Education* (pp. 539-548). Melbourne: Biomedical Multimedia Unit, The University of Melbourne.

Van Gelder, T. J., Bissett, M., & Cumming, G. (2004). Cultivating expertise in informal reasoning. *Canadian Journal of Experimental Psychology*, *58*(2), 142–152. https://doi.org/10.1037/h0085794

VanGundy, A., (2004). *101 activities for teaching creativity and problem solving*. Pfeiffer. San Francisco.

Vaugelade, C., Fuglestad, A. B., Goodchild, S., & Sriraman, B. (2012). Mediated action in teachers' discussions about mathematics tasks. *ZDM – Mathematics Education*, *44*(5), 677-689. https://doi.org/10.1007/s11858-012-0423-0

Velten, K. (2009). *Mathematical modelling and simulation: Introduction for scientists and engineers.* Wiley-VCH.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior*, *17*(2), 167–181. https://doi.org/10.1016/S0364-0213(99)80057-3

Verschaffel, L., Greer, B., & Corte, E. de. (2000a). *Making sense of word problems*. Swets & Zeitlinger Publishers.

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020b). Word problems in Mathematics Education: A survey. *ZDM – Mathematics Education*, *52*(1), 1–16. https://doi.org/10.1007/s11858-020-01130-4

Vorhölter, K., Kaiser, G. & Borromeo Ferri, R. (2014). Modelling in Mathematics Classroom Instruction: An Innovative Approach for Transforming Mathematics Education. In Y. Li, E. Silver, & S. Li. (Eds)., *Transforming Mathematics Instruction. Advances in Mathematics Education* (pp. 21–36). Springer. https://doi.org/10.1007/978-3-319-04993-9_3.

Vos, P. (2011). What is 'authentic' in the teaching and learning of mathematical modelling?. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. A. Stillman (Eds.), *Trends in Teaching and Learning of Mathematical Modelling. International Perspectives on the Teaching and Learning of Mathematical Modelling* (pp. 713–722). Springer. https://doi.org/10.1007/978-94-007-0910-2_68

Vos, P. (2015). Authenticity in extra-curricular mathematics activities: Researching authenticity as a social construct. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 105–113). Springer. https://doi.org/10.1007/978-3-319-18272-8_8

Voskoglou, M. (2007). A stochastic model for the modelling process. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, engineering and economics: proceedings from the twelfth International Conference on the Teaching of Mathematical Modelling and Applications* (pp. 149-157). Chichester: Horwood.

Wang, S., Wang, H., Xia, J., & Xu, X. (2024). Simulation optimization with non-stationary streaming input data. In *Proceedings of the 2024 Winter Simulation Conference* (pp. 3494–3505). IEEE. https://doi.org/10.1109/WSC57941.2024.10495234

Watson, A., & Ohtani, M. (2015b). Themes and Issues in Mathematics Education Concerning Task Design: Editorial Introduction. In A. Watson, & M. Ohtani (Eds) *Task design in* *mathematics education. New ICMI Study Series* (pp. 3–15). Springer. https://doi.org/10.1007/978-3-319-09629-2_1

Watson, A., & Ohtani, M. (Eds.). (2015a). *Task design in mathematics education: An ICMI Study 22*. Springer. https://doi.org/10.1007/978-3-319-09629-2

Weber, H., & Weber, M. (2020). When means of implementation meet Ecological Modernization Theory: A critical frame for thinking about the Sustainable Development Goals initiative. *World Development, 136*, 105129. https://doi.org/10.1016/j.worlddev.2020.105129

Wess, R., Klock, H., Siller, H.-S., & Greefrath, G. (2021). *Measuring professional competence for the teaching of mathematical modelling: A test instrument*. Springer. https://doi.org/10.1007/978-3-030-78071-5

Wilensky, U., & Rand, W. (2015). *An introduction to agent-based modeling: Modeling natural, social, and engineered complex systems with NetLogo*. MIT Press.

Winters, N. (2007). What is mobile learning. In M. Sharples (Ed.), *Big issues in mobile learning (pp. 7–11): LSRI University of Nottingham.*

Xenofontos, C., Alkan, S. H., & Andrews, P. (2022). Estimation in the primary mathematics curricula of Cyprus, Greece and Turkey: A privileged or prevented competence? *Athens Journal of Education, 9*, 1–22. https://doi.org/10.30958/aje.9-1-1

Xenofontos, C., Fraser, S., Priestley, A. & Priestley, M. (2021). Mathematics teachers and social justice: a systematic review of empirical studies. *Oxford Review of Education*, *47*(2), 135–151. https://doi.org/10.1080/03054985.2020.1807314

Yamamoto, G. T. (2013). *Mobile learning workshop report Turkey*. Okan University.

Zaidan, S., & El Fadel, M. (2024). Sustainable development goals in energy system models: A systematic interlinkages mapping analysis. *Journal of Cleaner Production, 464*, 142638. https://doi.org/10.1016/j.jclepro.2024.142638

Zienkiewicz, O. C., Taylor, R. L., & Zhu, J. Z. (2013). *The finite element method: Its basis and* fundamentals (7th ed.). Butterworth-Heinemann.