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Geometric Control Problems over Networks

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To Nika and my family.

Abstract

This thesis aims to bridge classical geometric control theory with the theory of structured systems within a network-based framework. The proposed methodology integrates structural and geometric control techniques and leverages the key observation that the subspaces of interest contain or are contained in a subspace generated by canonical basis vectors. Since these canonical vectors correspond to individual network nodes, the algebraic analysis of such subspaces can be directly translated into an analysis over node sets.

The first paper included in this thesis investigates the controllability of temporal networks, modeled as linear, piecewise-constant, time-varying dynamical systems. In particular, an upper bound on the number of snapshots, i.e., time intervals during which the system matrices remain constant, necessary to achieve controllability, is derived. The proof exploits structural properties by decomposing the controllability subspace associated with each snapshot into a component which varies depending on the specific entries of the system matrix, and a fixed component, independent of those entries.

The second paper introduces the minimal node cardinality disturbance decoupling problem, studied under three feedback paradigms: state feedback, output feedback, and dynamic feedback. The formulation and analysis of this problem rely on classical concepts from geometric control theory, such as controlled and conditioned invariant subspaces. By exploiting the fixed structure of these subspaces, they can be reformulated as sets of nodes, providing a network-based interpretation that enhances both intuition and visual interpretability.

The third paper applies the developed theoretical framework to the minimal input cardinality disturbance decoupling problem via output feedback in a system of coupled oscillators, linearized around a stable phase-locked synchronized state, and applies it to the disturbance decoupling of power grids. This application illustrates the practical relevance and versatility of the proposed approach.

Overall, the thesis shows that, under mild assumptions, complex geometric control structures can be represented in an intuitive and visually meaningful manner when analyzed through the lens of networks. This perspective not only facilitates the study of large-scale systems, but also opens new research directions at the intersection of geometric control and network theory.

Populärvetenskaplig sammanfattning

Denna avhandling undersöker hur idéer från reglerteknik, som handlar om hur man styr och kontrollerar komplexa system, kan förstås på ett nytt sätt genom att se dem ur ett nätverksperspektiv. I ett nätverk representerar varje nod en komponent av tillståndet, och kopplingarna, representerade av nätverkets länkar, visar hur dessa påverkar varandra över tid.

Arbetet börjar med att undersöka om hela nätverket kan styras till ett önskat tillstånd med hjälp av en insignal. Detta koncept kallas för en styrbarhetsstudie. Ett nätverk som förändras i tid kallas ett temporalt nätverk. Ett av huvudresultaten visar hur man i ett temporalt nätverk kan bestämma det minsta antal tidsintervall (eller "snapshots") av nätverket som krävs för att säkerställa att sådan styrning är möjlig.

Dessa idéer utvidgas därefter till ett relaterat koncept som kallas störningsundertryckning. Enkelt uttryckt handlar det om att utforma styrning så att oönskade störningar som påverkar en del av nätverket inte sprider sig till vissa målnoder. Studien introducerar metoder för att hitta det minsta antalet noder som behöver styras och/eller mätas för att uppnå denna frikoppling, beroende på vald styrstrategi.

Slutligen används de teoretiska verktygen för att dämpa störningar i ett system av kopplade oscillatorer, en vanlig modell för synkroniserat beteende inom fysik, biologi och elnät.

Sammantaget visar avhandlingen att många komplexa matematiska idéer inom geometrisk styrteori kan göras betydligt mer intuitiva och visuellt begripliga när de betraktas ur ett nätverksperspektiv.

Acknowledgments

I would like to sincerely thank my supervisor, Claudio Altafini, for introducing me to stimulating and challenging research topics, and for his patience and guidance in teaching me how to conduct research. I am also grateful to the Head of Division, Martin Enqvist, for fostering the welcoming and inspiring research environment that I have the privilege to be part of, and to Ninna Stensgård for her kind support with administrative matters.

I am thankful to the entire RT group for the respectful and supportive environment, and for their help during times of need. I would like to thank Carmela Bernardo for the inspiring discussions during the early stages of my research, and my colleagues Joel Wendin and Ahsan Razaq, who started this journey with me and with whom I share, on a daily basis, many steps of the challenging path of a PhD student.

I would like to express my deepest gratitude to my mother, my father, and my brother for their constant support and for encouraging my decision to move to Sweden to pursue my PhD. I am also deeply thankful to my girlfriend, Nika, for her unwavering presence, care, and emotional support.

Finally, I would like to thank my friends and all the people who believe in me and support me in working toward the successful completion of this important chapter of my life.

Linköping, January 2026
Luca Lebon

Contents

Part I

Background

1

Introduction

Geometric control theory has long been recognized as one of the most elegant and powerful frameworks for the analysis and synthesis of linear dynamical systems. Emerging in the 1960s and 1970s from the pioneering works of Basile, Marro, Wonham, and others, geometric control provided a unifying, coordinate-free language to describe fundamental control-theoretic notions. Unlike classical transfer-function-based methods, geometric control theory formulates control problems in terms of subspaces of the state space that are invariant under the system dynamics and contain or are contained in a given subspace. For example, the notion of target-nulling controlled invariant subspace characterizes a subspace that is contained in the kernel of a matrix (for us, the matrix of the targets) and can be rendered, in closed loop, invariant under the system dynamics. This abstraction yields a theory of remarkable conceptual clarity and internal consistency, making it possible to characterize and solve complex multivariable control problems. As an example, the Disturbance Decoupling Problem (DDP), that is, the problem of protecting in closed loop some target outputs from the influence of disturbance inputs entering the state equation, can be cast entirely in geometric control terms and its solvability depends on inclusion conditions between subspaces with properties of invariance in closed loop.

However, the reliance on vector space representations also poses challenges when the systems under investigation become large. Although the geometric framework remains mathematically sound regardless of the system dimension, its implementation requires the explicit computation of invariant and controlled invariant subspaces, a task that scales poorly with system size. Furthermore, classical geometric control theory traditionally assumes a numerical realization of the system matrices A, B, C, D. Such a realization may obscure structural properties, i.e., those qualitative features of a system that depend only on its interconnection topology

or sparsity pattern and not on the numerical values of its parameters. In many modern applications, particularly those involving large-scale dynamical systems, it is precisely these structural properties that capture the essence of the system's behavior.

This thesis aims to bridge the conceptual richness of geometric control theory with the structural systems approach, pioneered, among others, by Dion, Commault and Van der Woude, in a network-based setting. The overarching goal is to develop a framework that retains the rigor and expressiveness of geometric control while exploiting the inherent structure in networked systems. The central methodological insight is that the controllable and controlled-invariant subspaces associated with a networked dynamical system contain a fixed subspace generated by canonical basis vectors. Such subspaces are invariant of the entries of the system matrix A and depend only on the underlying network topology. Because each canonical vector corresponds to a distinct node of the network, this observation allows one to interpret the operations on subspaces directly as operations on sets of nodes. As a result, problems that are classically formulated in terms of linear algebraic conditions can be reinterpreted in terms of graph-theoretic and set-theoretic constructs, enabling novel insights and more scalable computational procedures. Moreover, properties of controlled and conditioned invariance, that are classically formulated in terms of vector subspaces, can be rewritten in terms of network node sets. This offers the possibility to design feedback laws directly from the node sets, leading to an intuitive graphical interpretation of otherwise abstract concepts.

This reinterpretation yields several significant new results. First, it enables the derivation of an upper bound on the minimum number of temporal snapshots required to guarantee structural controllability in a temporal network with given control inputs. This result directly links geometric notions of controllability with structural properties of the underlying network topology. Second, it provides a rigorous foundation for defining and analyzing new classes of network-oriented control problems, such as the minimal node cardinality disturbance decoupling problems, which seek the smallest set of control input nodes (and of measurement nodes) from which a feedback law that suppresses external disturbances can be computed explicitly.

More in detail, the thesis is organized around three main contributions, each presented in a separate paper.

The first paper focuses on the controllability of temporal networks, which are modeled as linear, piecewise-constant, time-varying systems. In this setting, the system matrices remain constant over preassigned time intervals, referred to as snapshots, but change their numerical entries between intervals. The paper establishes an upper bound on the number of snapshots required to ensure structural controllability, given a fixed set of control inputs, under the assumption that each time-invariant snapshot is not structurally controllable. The proof is based on a decomposition of the controllable subspace at each snapshot into two components: a subspace which varies depending on the specific numerical values of the system matrices, and a fixed subspace which depends only on the network structure. This

decomposition reveals a deep interplay between temporal variability and structural controllability in networked systems.

The second paper introduces and formalizes the minimal node cardinality disturbance decoupling problem. This problem extends the classical disturbance decoupling (DDP) theory, originally developed in the geometric control framework, to the context of networked systems. The problem is analyzed under three feedback paradigms: state feedback (minimal input node cardinality DDPSF), output feedback (minimal input/output node cardinality DDPOF), and dynamical feedback (minimal input/output node cardinality DDPDF). Depending on the approach that is used, several key geometric concepts that classically rely on subspaces are translated on sets of nodes. More precisely, the properties of controlled invariance and conditioned invariance, which are essential for the solvability of the disturbance decoupling problems, are written in terms of sets of nodes. The solvability of the disturbance decoupling via state feedback translates, in this setting, to the requirement that at least a control input must be present in each path from the disturbance nodes to the target nodes. When the full state is not measurable, the solvability of the disturbance decoupling via output feedback requires that a measurement node and control input node are placed at distance one in each path from the disturbance nodes to the target nodes. Finally, the solvability of the disturbance decoupling via dynamical feedback relaxes the requirement that measurement nodes and input nodes must be at distance one, by allowing a compensator to estimate the state values at the nodes that are contained within the measurement nodes and the control input nodes. Polynomial-time algorithms that are based on a modified version of the max-flow/min-cut algorithm are implemented to identify the minimal sets of control input nodes and of measurement nodes that fulfill the aforementioned characterizations. In summary, the approach on set of nodes allows to constructively identify the relevant nodes to be controlled and to be measured, from which it is possible to explicitly design the feedback matrices. This network-based reformulation not only improves the interpretability of geometric control constructs but also provides new algorithmic avenues for solving large-scale decoupling problems.

The third paper demonstrates the applicability of the proposed theoretical framework to a concrete test case: the minimal input cardinality disturbance decoupling problem via output feedback in a system of coupled oscillators. The oscillators are linearized around a stable phase-locked synchronized state, representing a common modeling scenario in network synchronization. By leveraging the developed geometric-structural framework, the paper provides an interesting application of disturbance decoupling in power network systems.

Overall, this thesis shows that complex geometric control structures can be recast in an intuitive and visually interpretable form when viewed through the lens of network theory. This geometric-structural synthesis not only enhances the conceptual understanding of classical control problems but also extends their applicability to large-scale systems. In doing so, it lays the groundwork for a unified perspective that connects geometric control theory, network science, and structural systems

theory, offering new directions for both theoretical research and practical applications in the control of complex dynamical networks.

1.1 Contributions

This thesis investigates geometric and structural control notions over networks, focusing on the graphical interpretation of concepts that are classically formulated on vector subspaces. Paper A focuses on the controllability of a special class of time-varying systems, called temporal networks, by exploiting the decomposition of the controllable subspace into a fixed subspace and a subspace that varies with the numerical entries of the system matrix. Paper B introduces the concepts of invariant node set, controlled invariant node set, and conditioned invariant node set, and exploits them in the formulation of a novel geometric control problem over networks, called the minimal node cardinality disturbance decoupling problem. Paper C applies the framework and the theoretical results of Paper B to a system of coupled oscillators linearized at synchronization and simulates the results on the power network benchmark IEEE New England 39-bus system.

The main contributions of the thesis are detailed below.

- The formal proof of the "fundamental advantage" of a temporal network. (Paper A)
- The partial characterization of the zero-measure set of realizations that do not increase controllability in a temporal network with fixed topology. (Paper A)
- A reinterpretation of controlled and conditioned A-invariance for sets of nodes. (Paper B)
- Sufficient graphical conditions for the solvability of DDPSF, DDPOF and DDPDF. (Paper B)
- The introduction of novel minimal node cardinality disturbance decoupling problems for state feedback, output feedback and dynamical feedback, and polynomial-time algorithms for finding the optimal solutions of the minimal node cardinality DDPSF and DDPDF. (Paper B)
- The derivation of closed-form formulas for the decoupling state feedback matrix, the output feedback matrix, and the dynamical compensator, in terms of the optimal solutions of the corresponding minimal node cardinality disturbance decoupling problems. (Paper B)
- An application of the graphical framework to the minimal node cardinality disturbance decoupling of coupled oscillators linearized at a stable synchronization state and a simulation of the results on the power network benchmark IEEE New England 39-bus system. (Paper C)

1.2 Thesis outline

Part I: Background

Part I introduces in detail the theoretical material used throughout the thesis. In Chapter 2, the preliminary material covers concepts and notions from graph theory, geometric control theory, and structured systems theory and offers a complete theoretical framework for the Papers included in Part II. More precisely, Section 2.1 contains the fundamental notions on network theory that are used throughout the thesis. Section 2.2 introduces the notion of controllability under both the geometrical and the structural points of view. In Subsection 2.2.1, in relation to the controllable and unobservable subspaces, the definition of A-invariant subspace and of A-invariant node set are given. In Subsection 2.2.2, the definition of structural controllability is given in great detail and in Subsubsection 2.2.2.1, an extensive literature review on the fixed controllable subspace is provided. In Section 2.3, the concepts of controlled and conditioned A-invariance are introduced both for subspaces and for sets of nodes. In Section 2.3.1, the controllable and unobservable subspaces $\mathcal{R}$, $\mathcal{Q}$, and their associated node sets $\mathcal{R}^\circ$, $\mathcal{Q}^\circ$, together with the maximal controlled invariant and minimal conditioned invariant subspaces $\mathcal{Z}^*$, $\mathcal{S}^*$, and their associated node sets $\mathcal{Z}^\circ$, $\mathcal{S}^\circ$, are discussed in their underlying lattice and semilattice structures. In Subsection 2.3.1.1, the relationship between the classical maximal controlled A-invariant subspace, its fixed subspace, and the maximal controlled A-invariant node set is discussed (and likewise for the minimal conditioned A-invariant subspace). Chapter 3 contains additional results extending the contents of Paper B. These should not be regarded as background material for Part II, but rather as a preliminary formulation of future research problems. In particular, in Section 3.1, some of the results of Paper B are extended to account for pole placement and stability.

Part II: Publications

Part II of this thesis contains three edited research papers listed below.

Paper A: On controllability of temporal networks

Edited version of the paper:

> Luca Claude Gino Lebon, Francesco Lo Iudice, and Claudio Altafini. On controllability of temporal networks. *European Journal of Control*, 80:101046, 2024.

Summary In this paper, it is shown that when the structure of the graph underlying the temporal network remains the same throughout each temporal snapshot while the (non-vanishing) edge weights vary, then the temporal network will be completely controllable almost always, even when its static counterpart is not. An upper bound on the number of snapshots needed to achieve controllability is provided and a partial characterization of the zero-measure set of non-controllable

temporal networks is presented.

Contribution and background The author of this thesis contributed with theoretical development, analytical proofs, implementation, numerical simulations, and preparation of the manuscript. The co-author Claudio Altafini provided problem formulation and conceptual guidance, feedback on proofs, and critical revisions of the manuscripts. The co-author Francesco Lo Iudice provided support with the mathematical formulation and critical revisions of the manuscript.

Paper B: Geometric control theory over networks: minimal node cardinality disturbance decoupling problems

Edited version of the paper:

> Luca Claude Gino Lebon and Claudio Altafini. Geometric control theory over networks: Minimal node cardinality disturbance decoupling problems. *arXiv preprint*, 2025. doi: 10.48550/arXiv.2510.16689. Under review.

Summary In this paper, it is shown how to formulate and solve disturbance decoupling problems over networks while choosing a minimal number of input and output nodes. Feedback laws that isolate and eliminate the impact of disturbance nodes on specific target nodes to be protected are provided using state, output, and dynamical feedback. When reformulated in terms of sets of nodes rather than subspaces, the controlled and conditional invariance properties admit a simple graphical interpretation. For state and dynamical feedback, the minimal input and output cardinality solutions can be computed exactly in polynomial time, via max-flow/min-cut algorithms. The synthesis of the feedback is also provided in closed form in all three cases.

Contribution and background The author of this thesis contributed the majority of the work, including theoretical development, analytical proofs, implementation, numerical simulations, and preparation of the manuscript. The co-author provided conceptual guidance and the original problem formulation, feedback on proofs, and critical revisions of the manuscript.

Paper C: Minimal input cardinality disturbance decoupling of coupled oscillators via output feedback with application to power networks

Edited version of the paper:

> Luca Claude Gino Lebon, Johan Lindberg, and Claudio Altafini. Minimal input cardinality disturbance decoupling of coupled oscillators via output feedback with application to power networks. 2025. Under review.

Summary This paper addresses the minimal input cardinality disturbance decoupling problem for a class of coupled oscillator networks, with a specific focus on systems linearized around a stable phase-locked synchronized state. The proposed theoretical framework is applied to the linearized swing dynamics of power grids operating near synchronization. In this context, the disturbance decoupling problem corresponds to isolating subsets of nodes from exogenous disturbances by means of batteries that can both add or withdraw active power. Numerical simulations carried out on the IEEE New England 39-bus system show that the proposed methodology not only yields a minimal actuator placement ensuring effective disturbance rejection, but also preserves the internal stability of the closed-loop system.

Contribution and background The author of this thesis contributed the majority of the work, including theoretical development, analytical proofs, implementation, numerical simulations, and preparation of the manuscript. The co-author Claudio Altafini provided conceptual guidance, feedback on proofs, and critical revisions of the manuscripts. The co-author Johan Lindberg provided support with the power network model and critical revisions of the manuscript.

Publications

Works published by the author of this thesis during his doctoral studies are listed below in chronological order. Publications indicated by a ⋆ are included in the thesis.

⋆Luca Claude Gino Lebon, Francesco Lo Iudice, and Claudio Altafini. On controllability of temporal networks. *European Journal of Control*, 80:101046, 2024.

Luca Claude Gino Lebon and Claudio Altafini. Optimal disturbance decoupling over networks via state feedback. In *2025 64th IEEE Conference on Decision and Control (CDC)*. IEEE, 2025.

⋆Luca Claude Gino Lebon and Claudio Altafini. Geometric control theory over networks: Minimal node cardinality disturbance decoupling problems. *arXiv preprint*, 2025. doi: 10.48550/arXiv.2510.16689. Under review.

⋆Luca Claude Gino Lebon, Johan Lindberg, and Claudio Altafini. Minimal input cardinality disturbance decoupling of coupled oscillators via output feedback with application to power networks. 2025. Under review.

2

Geometric and structural notions for linear systems

This chapter revisits fundamental concepts of control theory, including controllability, controlled invariance, and their dual notions, from geometric and structural perspectives, and reinterprets them within a network-based framework.

2.1 Network theory

Definition 2.1 (Directed graph and adjacency matrix). We consider a directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a set of n nodes and $\mathcal{E}$ a set of q edges over $\mathcal{V}$, with the convention that $(v_i, v_j) \in \mathcal{E}$ means $v_i \to v_j$, i.e., v_i is the tail and v_j is the head of the edge. The matrix $A \in \mathbb{R}^{n\times n}$ is the adjacency matrix associated with $\mathcal{G}$. The weight of the edge $(v_i, v_j) \in \mathcal{E}$ is interchangeably denoted by A_{ji} or a_{ji}. If $A = A^\top$ the graph is called undirected.

Since a digraph is identified by $(\mathcal{V}, \mathcal{E})$ or, equivalently, by A, we sometimes write $\mathcal{G} = \mathcal{G}(A)$ or $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ to identify the digraph $(\mathcal{V}, \mathcal{E}, A)$.

Definition 2.2 (Structured class). Denote by $\mathcal{A}$ the class of adjacency matrices of $\mathcal{G}$ having "support" in $\mathcal{E}$:

$$\mathcal{A} = \{A \in \mathbb{R}^{n\times n} \text{ s.t. } A_{ji} \neq 0 \text{ iff } (v_i, v_j) \in \mathcal{E} \text{ and } A_{ji} = 0 \text{ iff } (v_i, v_j) \notin \mathcal{E}\}.$$

Remark 2.1. A special matrix in $\mathcal{A}$ is the indicator matrix $\mathbb{A}$, i.e., the unweighted version of A whose entries are defined as $\mathbb{A}_{ij} = \mathbb{I}(A_{ij} \neq 0)$, with $\mathbb{I}(\,\cdot\,)$ denoting the indicator function, equal to 1 if the condition is true and 0 otherwise. $(\mathcal{V}, \mathcal{E}, \mathbb{A})$ is the unweighted graph associated to $\mathcal{G}$.

Definition 2.3 (Incidence matrix). The *incidence matrix* $\Delta \in \mathbb{R}^{n\times q}$ associated with the digraph $\mathcal{G}$ encodes the relationship between nodes and edges. Each column of Δ corresponds to an edge, and the nonzero elements of Δ are defined component-wise as:

$$\Delta_{ij} = \begin{cases} +1, & \text{if node } i \text{ is the head (sink) of edge } j, \\ -1, & \text{if node } i \text{ is the tail (source) of edge } j, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4 (Laplacian matrix). The *Laplacian matrix* of $\mathcal{G}$ is defined for a symmetric nonnegative A as

$$L = \operatorname{diag}\left(\left\{\sum_{j=1}^{n} a_{ij}\right\}_{i=1}^{n}\right) - A \in \mathbb{R}^{n\times n}.$$

Equivalently, if Δ is the incidence matrix and A collects the edge weights, then

$$L = \Delta \operatorname{diag}\big(\{a_{ij}\}_{(i,j)\in\mathcal{E}}\big)\Delta^{\top}.$$

For a connected digraph, $\ker(\Delta^{\top}) = \ker(L) = \operatorname{span}(\mathbf{1}_n)$.

Definition 2.5 (In-degree and out-degree). Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ and the associated indicator matrix $\mathbb{A}$, the in-degree of a node $i \in \mathcal{V}$ denotes the number of ingoing edges connected to i and is defined by $k_{\text{in}}(i) = \sum_{j=1}^{n} \mathbb{A}_{ij}$. The out-degree of a node $i \in \mathcal{V}$ denotes the number of outgoing edges from i and is defined by $k_{\text{out}}(i) = \sum_{j=1}^{n} \mathbb{A}_{ji}$.

Definition 2.6 (Terminal subset). Given a digraph $\mathcal{G}$, a subset $\mathcal{Z} \subset \mathcal{V}$ is called a terminal subset of nodes if there is no outgoing edge from $\mathcal{Z}$, i.e.,

$$\forall\, (v_i,\, v_j) \in \mathcal{E},\ v_i \in \mathcal{Z} \implies v_j \in \mathcal{Z}.$$

Definition 2.7 (Directed path, stem, cycle and bud). A directed path $\mathcal{P}$ from node v_i to node v_j is a sequence of distinct nodes $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ such that $(v_k, v_{k+1}) \in \mathcal{E}$ for all $k \in \{i, i+1, \ldots, j-1\}$: $\mathcal{P} = \{v_i, \ldots, v_j\}$, or $\mathcal{P}^{v_i,v_j}$ when it is necessary to specify the extremities of the path.
The number of edges in the path is called the *length* of the path.
A path $\mathcal{P}^{v_i,v_j}$ of maximal cardinality in $\mathcal{G}$ satisfying $k_{\text{in}}(v_i) = 0$, where $k_{\text{in}}(v_i)$ denotes the in-degree of node v_i, is called a *stem*.
A path satisfying $v_i = v_j$ is called *cycle*. We call a *bud* an elementary cycle with an additional edge that ends, but does not begin, in a node of the cycle.

Definition 2.8 (Dilation). A dilation is a motif of $\mathcal{G}$ in which a set of nodes, say

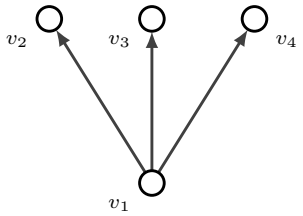


Figure 2.1: *Dilation with* $\mathcal{D} = \{v_2, v_3, v_4\}$ *and* $T(\mathcal{D}) = \{v_1\}$.

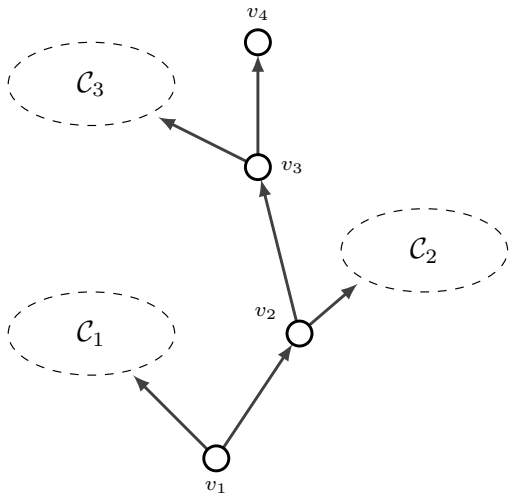


Figure 2.2: *Cactus with* $\mathcal{C}_i$ *cycle sets,* $i \in \{1, 2, 3\}$.

$\mathcal{D} \subset \mathcal{V}$, has parents $T(\mathcal{D}) = \{v_j \text{ s.t. } (v_j, v_i) \in \mathcal{E},\ v_i \in \mathcal{D}\}$ such that $|T(\mathcal{D})| < |\mathcal{D}|$. See Figure 2.1.

Definition 2.9 (Cactus graph). A *cactus* is a graph given by the union of a stem and buds. See Figure 2.2.

Definition 2.10 (Subspace associated with a set of nodes). Given a set of nodes $\mathcal{Z} \subseteq \mathcal{V}$, we can use $\mathcal{Z}$ to express subspaces of $\mathbb{R}^n$:

$$\mathcal{Z} \sim \{x \in \mathbb{R}^n \text{ s.t. } x_i \neq 0\ \forall\ i \in \mathcal{Z} \text{ and } x_i = 0\ \forall\ i \notin \mathcal{Z}\}\,.$$

Remark 2.2. We identify $\mathcal{Z}$ and the associated vector subspace, and sometimes write $x \in \mathcal{Z}$. Since a vector subspace contains the zero element, the identification "sets of nodes" $\sim$ "vector subspaces" holds almost always, except for a zero measure

set. This is the same principle behind the structural linear systems approach [1]. The only case in which this fact has to be made explicit is when we have equalities involving some matrix $A \in \mathcal{A}$, such as $Ax = y$, where $x \in \mathcal{X}$, $y \in \mathcal{Y}$ with $\mathcal{X}$, $\mathcal{Y}$ sets of nodes or vector subspaces. In this case, to avoid complications, we always replace A with the indicator matrix $\mathbb{A}$, and consider the expression $\mathbb{A}\mathcal{X} = \mathcal{Y}$. This is not needed when we have inclusions such as $A\mathcal{X} \subseteq \mathcal{Y}$.

For any $\mathcal{Z} \subset \mathcal{V}$, the notation $\mathcal{Z}^\perp$ is used both for the complement set $\mathcal{V} \setminus \mathcal{Z}$ and for the subspace orthogonal to (the vector space) $\mathcal{Z}$ in $\mathbb{R}^n$.

Remark 2.3. For each subspace $\mathcal{Z}$, there exists a full column rank matrix Z such that $\mathrm{Im}Z = \mathcal{Z}$ and $\ker Z = \{0\}$. Such matrix is called a *basis matrix* of $\mathcal{Z}$. When $\mathcal{Z}$ is a set of nodes, Z is a collection of elementary column vectors (one column for each node $v_i \in \mathcal{Z}$, having 1 in the v_i-th position and 0 elsewhere). As a consequence, the sum of vector subspaces and the set union can be identified: $\mathcal{Z}_1 + \mathcal{Z}_2 \sim \mathcal{Z}_1 \cup \mathcal{Z}_2$. Moreover, $Z^\top$ is full-row rank and $\ker Z^\top = (\mathrm{Im}Z)^\perp$. If $\mathcal{S} \subset \mathcal{V}$ denotes another set of nodes, then $\mathcal{S} \cap \ker Z^\top$ corresponds to the component of $\mathcal{S}$ orthogonal to $\mathrm{Im}Z$. In this setting, set subtraction and orthogonal complement coincide because all subspaces involved are spanned by disjoint subsets of orthonormal basis vectors. We therefore write $\mathcal{S} \cap \ker Z^\top \sim \mathcal{S} \setminus \mathrm{Im}Z$.

The column space of a matrix with elementary columns identifies a set of nodes in $\mathcal{G}$. In the following, the main sets of nodes used in the thesis are presented.

Definition 2.11 (Sets of nodes associated with a LTI system). Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, and the subsets of network nodes $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{T}$, of cardinality m, p, $|\mathcal{D}|$ and $|\mathcal{T}|$ respectively, it is possible to associate the following continuous-time linear time-invariant (LTI) dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw, \\ y &= Cx, \\ z &= Tx, \end{aligned} \tag{2.1}$$

where we have associated one state variable $x_i \in \mathbb{R}$ of the vector $x = [x_1 \dots x_n]^\top$ to each node v_i of the graph and chosen the matrices $B \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{n \times |\mathcal{D}|}$ with elementary columns, and the matrices $C \in \mathbb{R}^{p \times n}$ and $T \in \mathbb{R}^{|\mathcal{T}| \times n}$ with elementary rows, s.t. we obtain the following identifications

- $\mathcal{B} \cong \mathrm{Im}\, B$, called the **set of control inputs**;
- $\mathcal{C} \cong \mathrm{Im}\, C^\top = (\ker C)^\perp$, called the **set of measurements**;
- $\mathcal{D} \cong \mathrm{Im}\, D$, called the **set of disturbances**;
- $\mathcal{T} \cong \mathrm{Im}\, T^\top = (\ker T)^\perp$, called the **set of targets**.

Definition 2.12 (In- and out-boundaries). On $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, for any subset

of nodes $\mathcal{W} \subseteq \mathcal{V}$ we define the in- and out-boundaries as follows:

$$\partial_-(\mathcal{W}, A) := \{v_i \in \mathcal{W} \text{ s.t. } \exists\,(v_i, v_j) \in \mathcal{E},\, v_j \in \mathcal{W}^\perp\},$$

i.e., the set of nodes in $\mathcal{W}$ that have at least one outgoing edge leading to a node in $\mathcal{W}^\perp$. Similarly,

$$\partial_+(\mathcal{W}, A) := \{v_j \in \mathcal{W}^\perp \text{ s.t. } \exists\,(v_i, v_j) \in \mathcal{E},\, v_i \in \mathcal{W}\},$$

is the set of nodes in $\mathcal{W}^\perp$ that have at least one incoming edge from $\mathcal{W}$ (see Figure 2.3).

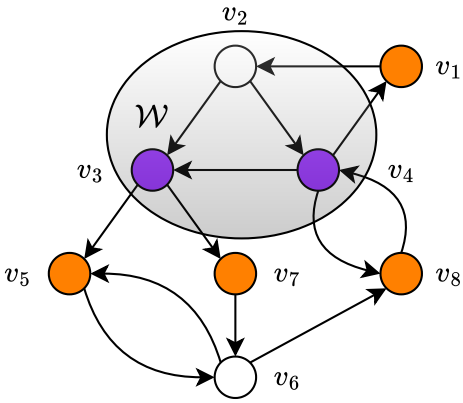


Figure 2.3: *Given $\mathcal{W} \subseteq \mathcal{V}$, its in-boundary $\partial_-(\mathcal{W}, A) = \{v_3, v_4\}$ is in purple, while its out-boundary $\partial_+(\mathcal{W}, A) = \{v_1, v_5, v_7, v_8\}$ is in orange.*

From the previous definition, it is obvious that $\partial_\pm(\mathcal{W}, A) = \partial_\mp(\mathcal{W}^\perp, A^\top)$. Notice that in general $\partial_+(\mathcal{W}, A) \subseteq A\partial_-(\mathcal{W}, A)$ and the inclusion can be strict.
In the context of disturbance decoupling, investigated in Paper B, the notions of in- and out- boundary are closely related to those of cut and cut-sets, given in the following.

Definition 2.13 (Cut). A cut (S, T) of $\mathcal{G}$ is a partition of the node set $\mathcal{V}$ into two subsets S and T. The *cut-set* (or *edge cut*) of (S, T) is defined as $\delta(S, T) := \{\,(u, v) \in \mathcal{E} \text{ s.t. } u \in S,\ v \in T\,\}$, i.e., the set of edges directed from S to T. In a weighted directed graph, the *value* of a cut is the sum of the weights of the edges in $\delta(S, T)$. If $\mathcal{G}$ is unweighted, the value of a cut (S, T) is the number of edges in $\delta(S, T)$. A cut is said to be *minimum* if its value is not larger than that of any other cut. An *s-t* cut, is a cut where $s \in S$ (called source) and $t \in T$ (called sink).

It is well known (see [2]) that the problem of finding a minimum *s-t* cut is the dual, in the linear programming sense, of the problem of maximizing the flow value from s to t. For completeness, we provide the following definition.

Definition 2.14 (Flow). A flow in $\mathcal{G}$ is a mapping $f : \mathcal{E} \to \mathbb{R}^+$ for the edge (u, v), denoted by f_{uv}, subject to the following constraints:

- *Capacity constraint*: $\forall\,(u,v) \in \mathcal{E},\ f_{uv} \leq c_{uv}$, where c_{uv} denotes the weight (or capacity) of the edge (u,v);
- *Conservation of flow*: $\forall\, v \in \mathcal{V} \setminus \{s,t\}$, where $s,t \in \mathcal{V}$ denote the source and sink, respectively, $\sum_{\{u\,\text{s.t.}\,(u,v)\in\mathcal{E}\}} f_{uv} = \sum_{\{w\,\text{s.t.}\,(v,w)\in\mathcal{E}\}} f_{vw}$.

The value of a flow f is defined as[1] $|f| = \sum_{\{v\,\text{s.t.}\,(s,v)\in\mathcal{E}\}} f_{sv} = \sum_{\{v\,\text{s.t.}\,(v,t)\in\mathcal{E}\}} f_{vt}$.

Theorem 2.1 (Max-flow/min-cut Theorem). *The value of the maximum flow in a network equals the value of any minimum s-t cut.*

There exist polynomial-time algorithms for computing minimum cuts [2–4]. In the context of disturbance decoupling, investigated in Paper B, the source node will be denoted by $v^{\mathcal{D}}$ and the sink by $v^{\mathcal{T}}$.

2.2 Controllability

This section provides background theory on controllability. Consider the continuous-time LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y &= Cx(t), \end{aligned} \tag{2.2}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. Fix a finite time $t > 0$ and set $x_0 = 0$. The state at time t produced by an input $u \in L^2([0,t];\mathbb{R}^m)$ is $x(t) = \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$. We define the *reachability operator* as the linear map $\mathcal{L}_t : L^2([0,t];\mathbb{R}^m) \to \mathbb{R}^n$ s.t. $x(t) = \mathcal{L}_t(u) = \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$. The *reachable subspace* at time t is $\mathcal{R}(t) := \operatorname{Im}\mathcal{L}_t$. The system is controllable on $[0,t]$ if $\mathcal{R}(t) = \mathbb{R}^n$. If we equip $L^2([0,t];\mathbb{R}^m)$ with the inner product $\langle u, w\rangle_{L^2} = \int_0^t u(\tau)^\top w(\tau)\,d\tau$ and $\mathbb{R}^n$ with the Euclidean inner product $\langle z, x\rangle_{\mathbb{R}^n} = z^\top x$, we can define the *reachability Gramian* $W_r(t) := \mathcal{L}_t\mathcal{L}_t^* = \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau$, with $\mathcal{L}_t^* : \mathbb{R}^n \to L^2([0,t];\mathbb{R}^m)$ being the adjoint operator satisfying $\langle \mathcal{L}_t(u), x\rangle_{\mathbb{R}^n} = \langle u, \mathcal{L}_t^*(x)\rangle_{L^2}$, $\forall\, u \in L^2$, $x \in \mathbb{R}^n$. Since $W_r(t)$ is a $n \times n$ symmetric positive semidefinite matrix, it has real nonnegative eigenvalues, therefore we have the following.

Proposition 2.1 ([5]). $\operatorname{Im} W_r(t) = \operatorname{Im}\mathcal{L}_t$, *consequently,* $W_r(t)$ *is positive definite if and only if* $\mathcal{R}(t) = \mathbb{R}^n$.

Remark 2.4. Since we deal with continuous-time linear systems, the definitions of *controllability*, i.e., $\forall\, x_0 \in \mathbb{R}^n\, \exists\, u$ s.t. $x(t) = 0$, and of *reachability*, i.e., if $x_0 = 0$, $\forall\, x_f \in \mathbb{R}^n\, \exists\, u$ s.t. $x(t) = x_f$, coincide [5]. The reason being that

[1] $|f|$ is the standard notation for the value of the flow, and it does not mean absolute value in the numerical sense.

the controllability Gramian $W_c(t) = \int_0^t e^{-A\tau} B B^\top e^{-A^\top \tau} d\tau$ and the reachability Gramian $W_r(t) = \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau$ are related by the congruent relation $W_r(t) = e^{At}\, W_c(t)\, e^{A^\top t}$, which preserves positive definiteness. Moreover, if $W_r(t)$ is positive definite, for any $x_f \in \mathbb{R}^n$, the control input $u := \mathcal{L}_t^*(W_r(t)^{-1} x_f) = B^\top e^{A^\top (t-\tau)} W_r(t)^{-1} x_f$, yields $\mathcal{L}_t(u) = W_r(t) W_r(t)^{-1} x_f = x_f$ and is the unique minimizer of the control energy $\int_0^t \|u(\tau)\|^2 \, d\tau$ [6].

It is well known that the controllability of the linear system (2.2) does not depend on the time variable, but only on the pair (A, B). In fact, from the series expansion of $e^{A\tau}$ and the Cayley-Hamilton theorem, we have the following.

Proposition 2.2. $\operatorname{Im} \mathcal{L}_t = \operatorname{Im} K$, *where* $K := [B \; AB \; \dots \; A^{n-1}B] \in \mathbb{R}^{n \times nm}$ *is called the* ***controllability matrix****.*

We can therefore enumerate the following equivalent conditions for the controllability (equiv. reachability) of the pair (A, B).

Theorem 2.2. *Consider the continuous time LTI system* (2.2)*. We say that the pair* (A, B) *is controllable whenever one of the following equivalent conditions is met:*

1. $W_r(t)$ *is positive definite;*
2. $K = [B \; AB \; \dots \; A^{n-1}B]$ *has full row rank;*
3. *The PBH matrix* $[A - \lambda I \; B]$ *has full row rank* $\forall\, \lambda \in \mathbb{C}$*;*
4. *Let* λ *and* v *be any eigenvalue and any corresponding left eigenvector of* A*, i.e.,* $v^* A = v^* \lambda$*, then* $v^* B \neq 0$*, where* v^* *is the conjugate transpose of* v*;*
5. *The eigenvalues of* $A - BF$ *can be freely assigned (with the restriction that the complex eigenvalues are in conjugate pairs) by a suitable choice of the feedback matrix* F*.*

Remark 2.5. It is important to note that the matrix pencil $A - \lambda I$ loses rank only when λ is an eigenvalue of A. Therefore, if v is a left eigenvector of A corresponding to the eigenvalue λ, the full row rank of the PBH matrix requires that such eigenvector is not perpendicular to B, as stated in item 4. In particular, if A is in canonical Jordan form, item 3 means that, for each eigenvalue λ_i of A the rows of B corresponding to the last rows in the Jordan blocks of eigenvalue λ_i are linearly independent. Therefore, if the system (2.2) has m inputs, this means that a necessary condition for the controllability of (A, B) is that in the Jordan form of A there can be at most m Jordan blocks corresponding to the same eigenvalue, that is, each eigenvalue can have geometric multiplicity at most equal to m. If $m = 1$, the controllability of (A, B) requires that in the Jordan form of A there must be only one Jordan block for each eigenvalue, i.e., that B is a cyclic vector for A [7]. This fact will be further developed in the following sections, when structural

controllability is introduced.

Remark 2.6 (Pole placement). By item 5, the controllability of the pair (A, B) is equivalent to the pole placement of the closed-loop system. To acknowledge that, it is important to note that, if (A, B) is controllable and b_i is a nonzero column of B, then by Heymann's Lemma (see Prop. 6.5.1. of [7]) there exists a matrix $M \in \mathbb{R}^{m\times n}$ s.t. $(A + BM, b_i)$ is controllable. Then $\exists T$ invertible s.t. $(T^{-1}(A + BM)T, T^{-1}b_i)$ is in controllable canonical form. Now note that (A, B) is controllable if and only if $(A + BF, B)$ is controllable, and that the controllable subspaces coincide $\forall F \in \mathbb{R}^{m\times n}$ (see Prop. 6.1.1. of [7]). Therefore, pole allocation is possible for the pair $(T^{-1}(A + BM)T - T^{-1}b_iF, T^{-1}b_i)$ from the single input $T^{-1}b_i$. In fact, assume $(A + BM, b_i)$ is controllable with controllability matrix K. Then the closed-loop system $(A + BM) - b_iF$ has the monic polynomial $p(s) = p_0 + p_1 s + \cdots + p_{n-1}s^{n-1} + s^n$ as characteristic polynomial, if

$$F = e_n^\top K^{-1} p(A). \tag{2.3}$$

The formula given in (2.3) is known as the Ackermann formula, and it requires the controllable pair (A, B) to be brought in the single-input controllable pair $(A + BM, b_i)$ first (which is always possible, by Heymann's Lemma). There exist however more robust and computationally efficient methods to place the poles in the closed loop of a controllable pair (A, B). In this regard, as discovered by Moore and Laub in [8], one way is to find a basis of right eigenvectors[2] for the matrix A and a basis of inputs $u_i = -Fv_i$. More precisely, the closed-loop eigenstructure assignment problem under state feedback reads $(A - BF)v_i = \lambda_i v_i$, that can be rewritten as $(A - \lambda_i I)v_i = BFv_i$. Now, if $u_i = -Fv_i$, by extending the reasoning to all the eigenvalues this is equivalent to

$$[A - \lambda I \; B] \begin{bmatrix} v \\ u \end{bmatrix} = 0. \tag{2.4}$$

The pole placement problem can be dualized for the observability subspace, and is further extended in the same paper [8] and extensively detailed in [9], to compute the friends for the eigenstructure assignment in the output-nulling subspaces (and their duals input-containing subspaces). Those definitions will be part of the next sections, but we anticipate that with output equation $y = Cx + Du$ the formulation is equivalent to that of equation (2.4), if we replace the PBH matrix with the Rosenbrock matrix pencil $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$.

[2] In Moore's paper [8] the matrix A is assumed nondefective/diagonalizable for simplicity of notation, but similar results extend to non-diagonalizable controllable (A, B)-pairs and one needs to consider the generalized right eigenvectors of A.

2.2.1 Geometric definition

In this section we focus on a geometrical definition of controllability, that involves the property of A-invariance, which will be crucial in the following sections.

Subspace properties

Definition 2.15 (A-invariant subspace). A subspace $\mathcal{Z} \subseteq \mathbb{R}^n$ is A-invariant when $A\mathcal{Z} \subseteq \mathcal{Z}$.

Remark 2.7 (From Chapter 3 of [10]). If Z is a matrix whose columns form a basis for an A-invariant subspace $\mathcal{Z}$ of dimension k, then there exists a matrix X s.t. $AZ = ZX$. Moreover, the similarity transformation $T = [Z\ T_2]$ is s.t. $T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, with $A_{11} \in \mathbb{R}^{k\times k}$. Note that a common choice is to take $T_2 = Z^\perp$, with $Z^\perp$ being a basis matrix for $\mathcal{Z}^\perp$. Given an A-invariant subspace $\mathcal{Z}$, the orthogonal complement $\mathcal{Z}^\perp$ s.t. $\mathbb{R}^n = \mathcal{Z} \oplus \mathcal{Z}^\perp$ is not A-invariant in general. A necessary and sufficient condition for $\mathcal{Z}^\perp$ being A-invariant is that there exists a solution X for the Sylvester equation $A_{11}X - XA_{22} = -A_{12}$. In such a case $\exists \hat{T}$ s.t. $\hat{T}^{-1}A\hat{T} = \text{blkdiag}(\hat{A}_{11}, \hat{A}_{22})$, with $\hat{A}_{11} \in \mathbb{R}^{k\times k}$.

W.r.t. the change of basis as in Remark 2.7, we give the following definitions of internal and external stability of an A-invariant subspace[3], from [11].

Definition 2.16 (Internal and external stability of an A-invariant subspace). An A-invariant subspace $\mathcal{Z}$ is **internally stable** if every state-trajectory that originates in $\mathcal{Z}$ lies completely in $\mathcal{Z}$ and converges to the origin as t approaches infinity, that is, if and only if A_{11} is Hurwitz. Similarly, $\mathcal{Z}$ is **externally stable** if every state-trajectory that originates out of $\mathcal{Z}$ converges to $\mathcal{Z}$ as t approaches infinity, that is, if and only if A_{22} is Hurwitz[4].

Theorem 2.3 (Minimal input-containing A-invariant subspace (Theorem 3.3.1 and Algorithm 3.2.1 of [10])). *The controllable subspace $\mathcal{R}$ of the pair (A, B) is the minimal A-invariant subspace containing* $\operatorname{Im} B$*, and can be computed in at most n steps, that is when $\mathcal{R}_{k+1} = \mathcal{R}_k =: \mathcal{R}$ for $k \leq n-1$,*

- $\mathcal{R}_0 = \operatorname{Im} B$;
- $\mathcal{R}_k = \operatorname{Im} B + A\mathcal{R}_{k-1}, \quad k \in \{1, \dots, n-1\}$.

[3] These definitions of stability should not be confused with the standard definitions of internal stability of a linear system (which requires the full state matrix A to be Hurwitz), and of external stability of a linear system, also called input-output stability (which requires the poles of the input-output transfer function to be stable).

[4] A state-trajectory x that originates out of $\mathcal{Z}$ means that its initial state can be written as $x_0 = \alpha x_{\mathcal{Z}} + \beta x_{\mathcal{Z}^\perp}$, with $\beta \neq 0$, where $x_{\mathcal{Z}}$ denotes a generic initial state in $\mathcal{Z}$ and $x_{\mathcal{Z}^\perp}$ denotes a generic initial state in $\mathcal{Z}^\perp$. Formally, this means that the equivalence class of x_0, $[x_0] = x_0 + \mathcal{Z} \in \mathbb{R}^n/\mathcal{Z}$, depends only on the component $x_{\mathcal{Z}^\perp}$. The matrix A_{22} corresponds to the canonical projection on the quotient space $\mathbb{R}^n/\mathcal{Z}$.

Remark 2.8. If $\mathcal{R} = \mathbb{R}^n$ the pair (A, B) is controllable. Moreover, if $\mathcal{R} \subset \mathbb{R}^n$ but $\mathcal{R}$ is externally stable, then the pair (A, B) is stabilizable (i.e., the uncontrollable part in the Kalman decomposition of (A, B) is Hurwitz).

Node set properties

Definition 2.17 (A-invariant set of nodes). A subset of the network nodes $\mathcal{Z} \subseteq \mathcal{V}$ is an A-invariant set of nodes if $\mathcal{Z}$ is a terminal subset of nodes.

Remark 2.9. Given a directed graph $\mathcal{G}$, an A-invariant set of nodes $\mathcal{Z}$ different from $\emptyset$ or $\mathcal{V}$ and its complement $\mathcal{Z}^\perp$, it follows that $\mathcal{Z}^\perp$ can be A-invariant only if $\mathcal{G}$ has more than one connected component.

Proposition 2.3 (Minimal input-containing A-invariant node set). *Given a subset of nodes $\mathcal{B}$, the minimal A-invariant set of nodes $\mathcal{R}^\circ$ containing $\mathcal{B}$ can be computed in at most n steps, that is when $\mathcal{R}_{k+1} = \mathcal{R}_k =: \mathcal{R}^\circ$ for $k \leq n-1$,*

- $\mathcal{R}_0 = \mathcal{B}$;
- $\mathcal{R}_k = \mathcal{R}_{k-1} \cup \{v_j \notin \mathcal{R}_{k-1}$ s.t. $\exists$ *at least one* $(v_i, v_j) \in \mathcal{E}$ *with* $v_i \in \mathcal{R}_{k-1}\}$, $k \in \{1, \dots, n-1\}$.

Proof: $\mathcal{B} \subseteq \mathcal{R}^\circ$ by construction. Assume by contradiction that $\mathcal{R}^\circ$ is not A-invariant, then there must exist at least one node v_i in $\mathcal{R}^\circ$, a node v_j in $\mathcal{R}^{\circ\perp}$ and an edge (v_i, v_j). This means that $\mathcal{R}^\circ$ is not a terminal set of nodes. Minimality is trivial by the construction of the scheme, while the convergence of the scheme in at most n steps is guaranteed by the fact that the sequence is non-decreasing and the total number of network nodes is bounded by n. ■

Remark 2.10. Since $\mathcal{R}^\circ$ is a set of nodes, when interpreted as a vector space, it is spanned by canonical basis vectors $e_i \sim v_i$, $\forall\, v_i \in \mathcal{R}^\circ$. The link with the so-called fixed controllable space, also spanned by canonical basis vectors, will be exposed in the following section, once structural controllability is defined. For now we note that the controllable subspace $\mathcal{R}$ and the minimal A-invariant set of nodes $\mathcal{R}^\circ$ are s.t. $\mathcal{R} \subseteq \mathcal{R}^\circ$. This is not in contradiction with the minimality of the set $\mathcal{R}^\circ$, since the nodes in $\mathcal{R}^\circ$ correspond to the minimal coordinates that are excited in the controllable subspace. However, as shown in the following example, a generic generator of the controllable subspace is, in general, a weighted sum of canonical basis vectors, and only a subspace of $\mathcal{R}$ (the fixed controllable subspace) has the property of having canonical basis vectors as generators.

Definition 2.18 (Irreducible pair). A pair (A, B) is called reducible if there exists an input-containing A-invariant subspace $\mathcal{Z}$ of dimension $k < n$ s.t. after the $\mathcal{Z}$-induced similarity transformation T, the pair assumes the form $T^{-1}AT =$

$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, and $T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, with $A_{11} \in \mathbb{R}^{k\times k}$ and $B \in \mathbb{R}^{k\times m}$. A pair (A, B) is irreducible if such a transformation does not exist [12].

Example 2.1

Consider the following network of 3 nodes given in Figure 2.4, and take the A-invariant set of nodes $\mathcal{Z} = \{v_2, v_3\}$.

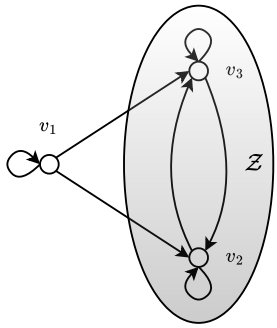


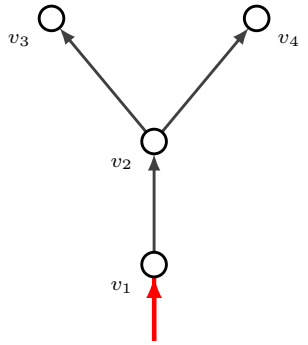
Figure 2.4: *Example of A-invariant set of nodes.*

One can choose the basis $Z = [e_2\ e_3]$ for $\mathcal{Z}$, thus $T = [Z\ e_1]$ gives the matrix $T^{-1}AT = \begin{bmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ 0 & 0 & a_{11} \end{bmatrix}$. Note that $\mathcal{Z}^{\perp} = \{v_1\}$ cannot be A-invariant since $\mathcal{G}$ has only one connected component, and that $\mathcal{Z}$ is internally stable iff $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ is Hurwitz, and externally stable iff $\Re(a_{11}) < 0$. Now observe that if $\mathcal{B} \subseteq \mathcal{Z}$, that is, if the A-invariant set is input-containing, then $T^{-1}B = [b_1\ b_2\ 0]^{\top}$ shows that $\mathcal{B}$ influences only the states in $\mathcal{Z}$. This transformation exists for any input-containing A-invariant subspace, and so it does for the controllable subspace. Therefore, unless $\mathcal{Z} = \mathcal{V}$, full state controllability cannot be reached and, w.r.t. the example, one would then focus on the controllable subspace of the pair $(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, [b_1\ b_2]^{\top})$.

Remark 2.11. We note that, as described in Remark 2.7, given an A-invariant set of nodes $\mathcal{Z}$ of cardinality k such similarity transformation exists. Moreover, if $\mathcal{Z}$ is input-containing then the pair (A, B) is in general reducible. A condition for the irreducibility of the pair (A, B) is that the minimal input-containing A-invariant set of nodes $\mathcal{R}^{\circ}$ has exactly cardinality n. This is equivalent to say that the graph $\mathcal{G}(A)$ is **input-connected**, i.e., for all $i \in \mathcal{V}$ either $i \in \mathcal{B}$ or in $\mathcal{G}$ there exists a directed path from a node $j \in \mathcal{B}$ to i. The set $\mathcal{R}^{\circ\perp}$ denotes the set of inaccessible nodes.

Example 2.2

Consider the following network with driver node in v_1, i.e., $\mathcal{B} = \{v_1\}$.



This is a directed acyclic graph, thus A is nilpotent (of order 3) regardless of its entries, that is for all A in the topologically induced class $\mathcal{A}$. Using the iterative schemes of Theorem 2.3 and of Proposition 2.3, we observe that

$$\mathcal{R} = \text{span}\{e_1, e_2, \alpha e_3 + \beta e_4\}, \quad \mathcal{R}^\circ = \{v_1, v_2, v_3, v_4\} = \mathcal{V}, \tag{2.5}$$

for some nonzero α and β dependent of the entries of A. This example shows that a necessary but not sufficient condition for controllability is that the graph is input-connected (equiv. the pair (A, B) is irreducible), that is when the sequence in Proposition (2.3) converges to $\mathcal{R}^\circ = \mathcal{V}$. It appears more clear now why $\mathcal{R} \subseteq \mathcal{R}^\circ$ if we realize that, due to the presence of a dilation, the third generator of $\mathcal{R}$ is a linear combination of the nodes v_3 and v_4.

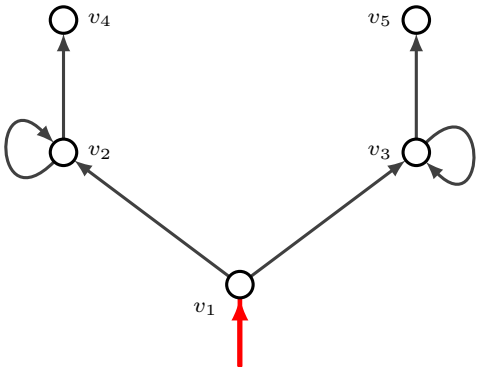
Remark 2.12. In Example 2.1 we wrote $\mathcal{R} = \text{span}\{e_1, e_2, \alpha e_3 + \beta e_4\}$, without specifying the nonzero constants α and β, explicitly highlighting the coordinates that are excited in the columns of K (i.e., $\{v_1\}$, $\{v_2\}$ and $\{v_3, v_4\}$) which correspond to the node "supports" for the generators of $\mathcal{R}$. Still, a clarification is necessary here since the coefficients involved in such generators depend on the entries of A and, more precisely, on the underlying topological structure of $\mathcal{G}$. In fact, the ij-th element of the generic block $A^\ell B$, say w_{ij}^ℓ, is the sum of the weights of the p-th of P paths $\pi_{ij}^\ell(p)$ from the j-th node in $\mathcal{B}$ to the i-th network node. Recalling that in a path of length ℓ from the j-th input node to the i-th node there is a sequence of edges $\{(r_1, r_2), (r_2, r_3), \ldots, (r_{\ell-1}, r_\ell)\}$ where node r_1 is the j-th element of $\mathcal{B}$ while $r_\ell = i$, we get

$$\pi_{ij}^\ell(p) = \prod_{h=1}^{\ell-1} a_{r_{h+1}, r_h}, \tag{2.6}$$

and thus

$$w_{ij}^\ell = \sum_{p=1}^{P} \pi_{ij}^\ell(p).$$

There can be situations where, independent of the entries of A, linear dependencies on the columns of K can arise due to, for example, a repeated factor in the product (2.6). As an example, consider the following network with $\mathcal{B} = \{v_1\}$.



Writing $\mathcal{R} = \text{span}\{e_1, \alpha e_2+\beta e_3, \gamma e_2+\delta e_3+\lambda e_4+\xi e_5, \gamma' e_2+\delta' e_3+\lambda' e_4+\xi' e_5, \gamma'' e_2+\delta'' e_3+\lambda'' e_4+\xi'' e_5\}$ without explicitly expressing the nonzero coefficients in terms of the entries of A might suggest that there exist three linearly independent vectors that excite the coordinates associated with the nodes in $\{v_2, v_3, v_4, v_5\}$. However, this is not the case due to the cycles in v_2 and v_3. The controllability matrix K is

$$\begin{bmatrix} e_1 & Ae_1 & A^2e_1 & A^3e_1 & A^4e_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & a_{21}a_{22} & a_{21}a_{22}^2 & a_{21}a_{22}^3 \\ 0 & a_{31} & a_{31}a_{33} & a_{31}a_{33}^2 & a_{31}a_{33}^3 \\ 0 & 0 & a_{21}a_{42} & a_{21}a_{22}a_{42} & a_{21}a_{22}^2a_{42} \\ 0 & 0 & a_{31}a_{53} & a_{31}a_{33}a_{53} & a_{31}a_{33}^2a_{53} \end{bmatrix}.$$

It easy to see that it has rank 4 for almost all the choices of entries of A. One notices that the last 3 columns are linearly dependent, in fact $A^4e_1 = \alpha A^2e_1 + \beta A^3e_1$, with $\alpha = -a_{22}a_{33}$ and $\beta = a_{22} + a_{33}$. The topological structure of $\mathcal{G}$ plays a crucial role in determining the dimension of the controllable subspace, as discussed in the next section.

We conclude this section with the dual definitions of the controllable subspace, namely, the unobservable subspace. This is done mainly to build and analogy with the concepts of controlled and conditioned invariance, which naturally extend the one of A-invariance, and which will be introduced in Section 2.3. In particular, the maximal controlled invariant set and the minimal conditioned invariant set are the natural extensions of the unobservable set and of the controllable set, respectively. From system (2.2), we have the following.

Theorem 2.4 (Maximal output-nulling A-invariant subspace (Corollary 3.3.1 and Algorithm 3.2.2 of [10])). *The unobservable subspace $\mathcal{Q}$ of the pair (C, A) is the maximal A-invariant subspace contained in* $\ker C$*, and can be*

computed in at most n steps, that is when $\mathcal{Q}_{k+1} = \mathcal{Q}_k =: \mathcal{Q}$ *for* $k \leq n-1$,

- $\mathcal{Q}_0 = \ker C$;
- $\mathcal{Q}_k = \ker C \cap A^{-1}\mathcal{Q}_{k-1}, \quad k \in \{1, \dots, n-1\}$.

Remark 2.13. The term $A^{-1}\mathcal{Q}_{k-1}$ denotes the preimage of A of the subspace $\mathcal{Q}_{k-1}$ which is also well defined when A is not invertible and corresponds to $A^{-1}\mathcal{Q}_{k-1} = \{x \in \mathbb{R}^n \text{ s.t. } Ax \in \mathcal{Q}_{k-1}\}$. The unobservable subspace of the pair (C, A) is the set of all the initial states that cannot be recognized from the output function in any finite time interval, therefore if $\mathcal{Q} = \{0\}$ the pair (C, A) is observable. Moreover, if $\mathcal{Q} \neq \{0\}$ but $\mathcal{Q}$ is internally stable, then the pair (C, A) is detectable.

As C is composed of elementary rows we have the identification $\ker C \cong \mathcal{V} \setminus \mathcal{C}$, hence in this case as well, we can derive the corresponding node-set formulation.

Proposition 2.4 (Maximal output-nulling A-invariant node set). *Given a subset of nodes* $\mathcal{C}$*, the maximal* A*-invariant set of nodes* $\mathcal{Q}^\circ$ *contained in* $\mathcal{V} \setminus \mathcal{C}$ *can be computed in at most n steps, that is when* $\mathcal{Q}_{k+1} = \mathcal{Q}_k =: \mathcal{Q}^\circ$ *for* $k \leq n-1$,

- $\mathcal{Q}_0 = \mathcal{V} \setminus \mathcal{C}$;
- $\mathcal{Q}_k = \mathcal{Q}_{k-1} \setminus \{v_i \in \mathcal{Q}_{k-1} \text{ s.t. } \exists \text{ at least one } (v_i, v_j) \in \mathcal{E} \text{ with } v_j \notin \mathcal{Q}_{k-1}\}$, $k \in \{1, \dots, n-1\}$.

The following duality result holds for both controllable and unobservable subspaces, and minimal input-containing A-invariant and maximal output-nulling A-invariant node sets.

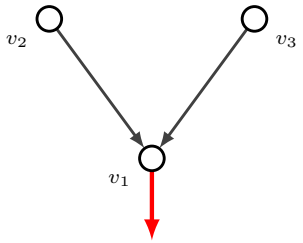
Proposition 2.5 (Duality - $\mathcal{R}, \mathcal{Q}$ and $\mathcal{R}^\circ, \mathcal{Q}^\circ$). *Denoting by* $\mathcal{R} = \mathcal{R}(A, B)$ *and* $\mathcal{R}^\circ = \mathcal{R}^\circ(A, \mathcal{B})$ *the controllable subspace and the minimal input-containing* A*-invariant node set, respectively, and by* $\mathcal{Q} = \mathcal{Q}(C, A)$ *and* $\mathcal{Q}^\circ = \mathcal{Q}^\circ(\mathcal{C}, A)$ *the unobservable subspace and the maximal output-nulling* A*-invariant node set, respectively, the following dualities hold*

- $\mathcal{Q}^\perp = \mathcal{R}(A^\top, C^\top), \quad \mathcal{R}^\perp = \mathcal{Q}(B^\top, A^\top)$;
- $\mathcal{Q}^{\circ\perp} = \mathcal{R}^\circ(A^\top, \mathcal{C}), \quad \mathcal{R}^{\circ\perp} = \mathcal{Q}^\circ(\mathcal{B}, A^\top)$.

We conclude this section with an example that highlights the relationship between $\mathcal{Q}$ and $\mathcal{Q}^\circ$ by explicitly computing the unobservable subspace using Theorem 2.4.

Example 2.3

Consider the following network with measurement node in v_1, i.e., $\mathcal{C} = \{v_1\}$.



Since there exists a path from any node to $\mathcal{C}$ we get $\mathcal{Q}^\circ = \emptyset$. However, using the recursion in Theorem 2.4 the unobservable subspace $\mathcal{Q}$ does not contain only the zero vector. In fact, $\ker C = \text{span}\{e_2, e_3\}$, and given a generic vector $x \in \mathbb{R}^3$ we have $Ax = [a_{12}x_2 + a_{13}x_3 \quad 0 \quad 0]^\top$. Therefore, $A^{-1}\ker C$ means requiring $Ax \in \text{span}\{e_2, e_3\}$, i.e., that $a_{12}x_2 + a_{13}x_3 = 0$. Therefore $A^{-1}\ker C = \text{span}\{e_1, a_{13}e_2 - a_{12}e_3\}$, hence $\mathcal{Q}_1 = \ker C \cap A^{-1}\ker C = \text{span}\{a_{13}e_2 - a_{12}e_3\}$, since $e_1 \notin \text{span}\{e_2, e_3\}$. Now note that $A^{-1}\mathcal{Q}_1$ means requiring $Ax \in \text{span}\{a_{13}e_2 - a_{12}e_3\}$, i.e., that $[a_{12}x_2 + a_{13}x_3 \;\; 0 \;\; 0]^\top = [0 \;\; \gamma a_{13} \;\; -\gamma a_{12}]^\top$ for some $\gamma \in \mathbb{R}$, which is satisfied for $\gamma = 0$ if, again, $a_{12}x_2 + a_{13}x_3 = 0$. Therefore, $A^{-1}\mathcal{Q}_1 = A^{-1}\ker C$ and $\mathcal{Q}_2 = \ker C \cap A^{-1}\mathcal{Q}_1 = \mathcal{Q}_1$ and the sequence stops. The unobservable subspace is thus $\mathcal{Q} = \text{span}\{a_{13}e_2 - a_{12}e_3\} \supseteq \mathcal{Q}^\circ$. Using Proposition 2.5 one notes that on the transposed graph, the orthogonal complement $\mathcal{Q}^\perp = \text{span}\{e_1, a_{12}e_2 + a_{13}e_3\}$ corresponds to the controllable subspace of the pair $(A^\top, C^\top)$. This example shows that as a dilation is a constraint for controllability, a transposed dilation is a constraint for observability.

2.2.2 Structural definition

Now that the controllable subspace $\mathcal{R}$ has been introduced and so is the minimal input-containing A-invariant set of nodes $\mathcal{R}^\circ$, we show that the controllable subspace can be decomposed as the sum of two subspaces, one that is independent of the entries of A, which in the literature is called the *fixed controllable subspace* [13], and one that "rotates" (its dimension is generically constant) with the entries of A. In this section, we take a generic matrix A in the topological class $\mathcal{A}$ (whose entry a_{ji} is either zero if there is no edge (v_i, v_j) or is an independent parameter, i.e., not related by algebraic equations with the other entries of A) and consider the input matrix B with elementary columns identified by the input set $\mathcal{B}$. The pair (A, B), when A is a generic matrix in $\mathcal{A}$, is called a structured pair, and the system (2.2) is called structured when considered as a function of the free parameters of $A \in \mathcal{A}$.

Definition 2.19 (Structural controllability). The structured system $\dot{x} = Ax + Bu$, with generic $A \in \mathcal{A}$ and $\text{Im}\, B = \mathcal{B}$, is structurally controllable if there exist values of the independent parameters for which it is controllable [14].

Remark 2.14. Controllability of a structured system is a generic property, that is, it holds true for all the values of the independent parameters except for a proper

algebraic variety in the parameter space. In the literature, the definition of structural controllability can be further extended to account for the cases in which a structured pair (A, B) is structurally controllable for all the values of the independent parameters. The pair (A, B) is called strongly structurally controllable in this case [15, 16].

Definition 2.20 (Generic rank). The generic rank of a structured matrix $A \in \mathcal{A}$, denoted by $|A|_g$, is the maximal rank that A can achieve as a function of all the free parameters.

Definition 2.21 (Dimension of the controllable subspace). The dimension of the controllable subspace of a structured system corresponds to $d := |K|_g$, where K is the controllability matrix of the structured pair (A, B), $A \in \mathcal{A}$.

Remark 2.15. With a slight abuse of notation, we denote the generic dimension of the subspace $\mathcal{R}$ by $d := |\mathcal{R}|_g$, meaning that $d = \dim(\mathcal{R})$ for almost all realizations $A \in \mathcal{A}$. In the proper variety where the dimension of the subspace $\mathcal{R}$ is different from d, we have $\dim(\mathcal{R}) < d$, by definition of generic rank.

Theorem 2.5 (Theorem 1 of [17]). *The dimension of the controllable subspace of a structured system is generically constant. Moreover, d is equal to the cardinality of the maximal disjoint union of cycles and stems rooted in the input nodes in $\mathcal{G}$.*

Theorem 2.6 ([14], [18]). *The structured pair (A, B) is structurally controllable if and only if*

1. *The graph $\mathcal{G}$ is input-connected;*
2. *The graph $\mathcal{G}$ contains no dilations, or equivalently, the PBH matrix $[A\ B]$ evaluated at the zero eigenvalue is generically full row rank.*

Remark 2.16. In the single input case, this directly translates in: "the structured pair (A, b) is structurally controllable if and only if $\mathcal{G}$ is spanned by a cactus rooted in b", where "is spanned by" means that there exists a cactus subgraph of $\mathcal{G}$ whose controlled root is directly connected to all the nodes in $\mathcal{G}$.

Condition 2 of Theorem 2.6 tells that the zero eigenstructure depends intrinsically on the existence of dilations in the graph. The fact that, for an irreducible structured pair (A, B), the modes corresponding to nonzero eigenvalues are generically controllable, is proven in the following result.

Theorem 2.7 (Theorem 2 of [12]). *Assume that the structured pair (A, B) is irreducible and denote with n_A the number of free parameters of $A \in \mathcal{A}$ and with*

n_B the number of free parameters of B[5]*. Then every mode of the structured system (A, B) that corresponds to the nonzero eigenvalues of A is controllable generically. More precisely, there exists a proper variety $V \subset \mathbb{R}^{n_A} \times \mathbb{R}^{n_B}$ such that for any $[A\ B] \in V^{\perp}$ the following holds*

- *A has $\nu(A)$ nonzero mutually distinct eigenvalues, where $\nu(A)$ denotes the largest order of a principal minor of A which is not identically zero;*
- *All the modes that correspond to these eigenvalues are controllable, i.e., $[A - \lambda I\ B]$ is full row rank, where λ denotes a nonzero eigenvalue of A.*

Lemma 2.1 (Lemma 2 of [12]). *Let $A \in \mathcal{A}$ and assume $\nu(A) > 0$. Then A has generically $\nu(A)$ nonzero eigenvalues which are mutually distinct and the other $n - \nu(A)$ eigenvalues are fixed at zero.*

Remark 2.17. As pointed out in Remark 2.5, from Lemma 2.1 the number of Jordan blocks that determine the minimal number of driver nodes that are necessary to achieve structural controllability only depend on the zero eigenvalue. In [12] it has been proven that, if a structured pair (A, B) is structurally controllable, the Jordan canonical form of (A, B) remains unaffected, in the generic sense, under the variation of the free parameters of (A, B). It is possible, for some classes of networks, to know exactly how the positions of cycles and dilations determine the Jordan form of A. An interesting example is given in [19, 20], for the class of graphs $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that have either $k_{\text{in}}(v_i) \leq 1$ or $k_{\text{out}}(v_i) \leq 1$, $\forall\, v_i \in \mathcal{V}$. Note that in general, the position of the cycles within the network, and the presence of "redundant" dilations may affect the geometric multiplicity of the zero eigenvalue, w.r.t. the results shown in [19, 20]. For a comprehensive study to quantify the degeneracy induced by the dilations, one would need to introduce the concepts of dilation delta, minimal dilation, and dilation choice sets [21, 22].

2.2.2.1 The fixed controllable subspace

Theorem 2.5 tells us that the generic dimension of the controllable subspace is fixed. However, if A_1 and A_2 are two different realizations within the same structured class $\mathcal{A}$, the controllable subspaces $\mathcal{R}(A_1, B)$ and $\mathcal{R}(A_2, B)$ of the pairs (A_1, B) and (A_2, B) are different in general. In [13] it has been proven that there exists a subspace of the controllable subspace, called fixed controllable subspace and denoted by $\mathcal{R}^g$, that is shared by almost all the realizations within the same structured class. That is, given the structured pairs (A_1, B) and (A_2, B), with $A_1, A_2 \in \mathcal{A}$, it holds $\mathcal{R}(A_1, B) = \mathcal{R}^g + \mathcal{R}_1^r$ and $\mathcal{R}(A_2, B) = \mathcal{R}^g + \mathcal{R}_2^r$, where $\mathcal{R}_i^r$ denotes the parameter-dependent complement of $\mathcal{R}^g$ in $\mathcal{R}(A_i, B)$, $i \in \{1, 2\}$. More precisely, we give the following definition.

[5] Since by our assumption B has m linearly independent elementary columns, $n_B = m$ is the number of driver nodes.

Definition 2.22 (Fixed controllable subspace). Given a structured class $\mathcal{A}$ and the input matrix B, we call fixed controllable subspace $\mathcal{R}^g$ the intersection of the controllable subspaces $\mathcal{R}_i$ for almost all the realizations $A_i \in \mathcal{A}$ [13].

The inclusion $\mathcal{R}^g \subseteq \mathcal{R} \subseteq \mathcal{R}^\circ \subseteq \mathbb{R}^n$ follows. We show in the next theorem that also $\mathcal{R}^g$ is generated by canonical basis vectors, and thus it can be thought of as a set of nodes (a subset of the nodes in $\mathcal{R}^\circ$), although in general not input-containing nor A-invariant.

Theorem 2.8 (Theorem 3 of [13]). *$\mathcal{R}^g$ is generated by canonical basis vectors in the state space defined as follows: the vector $e_i \in \mathcal{R}^g$ if and only if enlarging B with e_i the generic dimension of $\mathcal{R}$ does not change.*

As a consequence of Theorem 2.8, we have the following corollary.

Corollary 2.1. *If the vector $e_i \in \mathcal{R}$, then $e_i \in \mathcal{R}^g$ for almost all values of the nonzero entries in $A \in \mathcal{A}$. As a consequence, the generic dimension of $\mathcal{R}^g$, denoted by $\gamma =: |\mathcal{R}^g|_g$, is also generically fixed.*

Remark 2.18. As a consequence of Theorem 2.8, the fixed subspace $\mathcal{R}^g$ is spanned by $\{e_i,\ i \in \Lambda\}$, where Λ is a subset of $\mathcal{R}^\circ$ of cardinality γ. Thus, given a realization $A_k \in \mathcal{A}$, all the generators of $\mathcal{R}_k^r$, the complement of $\mathcal{R}^g$ in $\mathcal{R}(A_k, B)$, have nonzero projection on more than one vector $e_j,\ j \in \Lambda^\perp$. Under the assumption of irreducibility of the pair (A_k, B), i.e., when $\mathcal{R}^\circ = \mathcal{V}$, each generator of $\mathcal{R}_k^r$ results from the control signals propagating through a dilation.

Identifying the nodes of $\mathcal{R}^g$ can be done recursively using Theorems 2.8 and 2.5. However, this can be quite challenging for large networks. Under some assumptions, it is possible to identify the nodes that are in $\mathcal{R}^g$, as shown in the following results. Other recent studies on the fixed controllable nodes determination on directed acyclic graphs can be found in [23]. We give the following definition, and we note that the nodes in a bud satisfy such definition.

Definition 2.23 (Zero out-degree cycle). A cycle $\mathcal{C}_i$ in $\mathcal{G}$ is a zero out-degree cycle, denoted by $\mathcal{C}_i^{\text{no}}$, if none of the nodes in $\mathcal{C}_i^{\text{no}}$ is involved in a directed path towards a node that is outside of $\mathcal{C}_i^{\text{no}}$.

Proposition 2.6 (Theorem 1 and 2 of [24]). *The following are true*

- *If $v_i \in \mathcal{C}_i^{\text{no}}$, then $v_i \in \mathcal{R}^g$;*
- *If $\mathcal{R} = \mathcal{R}^\circ$, then $\mathcal{R}^g = \mathcal{R}$.*

As discovered in [13], if the structured pair (A, B) is irreducible and $d = |[A\ B]|_g$, it is possible to fully identify the nodes that are in $\mathcal{R}^g$. This characterization relies on the Dulmage-Mendelsohn (DM) decomposition of a bipartite graph [25] that

can be constructed from the structured pair (A, B). For this construction, it is required to extend the network nodes by duplicating all the network nodes, i.e., by creating the two node sets $\mathcal{X}^-$ and $\mathcal{X}^+$ with $|\mathcal{X}^-| = |\mathcal{X}^+| = |\mathcal{V}| = n$. Moreover, it is required to also include the input-nodes as an independent set of nodes. Since we assumed that the matrix B has m elementary columns, this simply corresponds to creating a duplicated set of the nodes in $\mathcal{B} \subseteq \mathcal{V}$, which we call $\mathcal{U}$, whose nodes are in one-to-one correspondence with the nodes in $\mathcal{B}$ and s.t. $|\mathcal{U}| = |\mathcal{B}| = m$. The bipartite graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ associated with the structured pair (A, B), $A \in \mathcal{A}$, is defined as follows.

Definition 2.24 (Bipartite graph of the structured pair (A, B)). We call bipartite graph of the structured pair (A, B) the bipartite graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$, with $\overline{\mathcal{V}} = \mathcal{V}^+ \cup \mathcal{V}^-$, where $\mathcal{V}^+ = \mathcal{X}^+ \cup \mathcal{U}$, $\mathcal{V}^- = \mathcal{X}^-$, with $\mathcal{X}^+ = \{x_1^+, \dots, x_n^+\}$, $\mathcal{X}^- = \{x_1^-, \dots, x_n^-\}$, and $\mathcal{U} = \{u_1, \dots, u_m\}$, and with edge set $\overline{\mathcal{E}} = \mathcal{E}_A \cup \mathcal{E}_B$, s.t. $\mathcal{E}_A = \{(x_j^+, x_i^-) \text{ s.t. } A_{ij} \neq 0\}$ and $\mathcal{E}_B = \{(u_j, x_i^-) \text{ s.t. } B_{ij} \neq 0\}$.

The total number of nodes in $\overline{\mathcal{G}}$ is $|\overline{\mathcal{V}}| = 2n + m$, and under the assumption that B has elementary columns, i.e., that $\operatorname{Im} B \cong \mathcal{B}$, then $|\overline{\mathcal{E}}| = q + m$, where $q = |\mathcal{E}|$.

Definition 2.25 (Matching). A matching in a bipartite graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ is an edge set $\mathcal{M} \subseteq \overline{\mathcal{E}}$ such that the edges in $\mathcal{M}$ have no common node. A matching $\mathcal{M}$ is called *maximum* if its cardinality is maximum. A matching $\mathcal{M}$ is called *perfect* if it covers all the nodes in $\overline{\mathcal{G}}$.

Proposition 2.7 (Proposition 1 of [13]). *Given a structured pair (A, B) with associated bipartite graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$, then $|[A\ B]|_g$ is equal to the cardinality of a maximum matching in $\overline{\mathcal{G}}$. In particular, $|[A\ B]|_g = n$ iff $\exists\, \mathcal{M}$ with $|\mathcal{M}| = n$.*

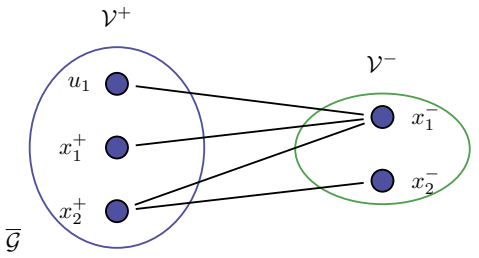
Remark 2.19. For the specific structure of the bipartite graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ a matching $\mathcal{M}$ of cardinality n is perfect and therefore it is also maximum.

From Theorem 2.6, if there exists a perfect matching in $\overline{\mathcal{G}}$, the structured pair (A, B) to be uncontrollable must be reducible (equiv., $\mathcal{R}^\circ \subset \mathcal{V}$), as shown in the following example.

Example 2.4

Consider the following network with $\mathcal{B} = \{v_1\}$ and with an inaccessible node v_2,

from which we can associate the bipartite graph $\overline{\mathcal{G}}$ given below.



There exists a maximum matching of size $n = 2$, meaning that the structured matrix

$$[A\ B] = \begin{bmatrix} * & * & 1 \\ 0 & * & 0 \end{bmatrix}$$

has generic rank $2 = n$. However, the network is not input-connected, which reflects the fact that the structured pair (A, B) is reducible.

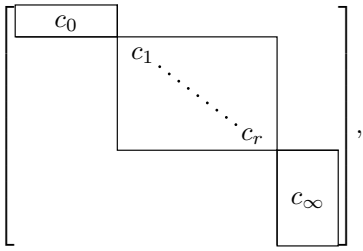
It is possible to parameterize all the maximum matchings in a bipartite graph using the DM-decomposition. The DM-decomposition of the bipartite graph $\overline{\mathcal{G}} = (\mathcal{V}^+ \cup \mathcal{V}^-, \overline{\mathcal{E}})$ is the uniquely defined family of bipartite subgraphs $\overline{\mathcal{G}}_i = (\mathcal{V}_i^+ \cup \mathcal{V}_i^-, \overline{\mathcal{E}}_i)$, $i \in \{0, 1, \ldots, r, \infty\}$, called the DM-components, where $(\mathcal{V}_0^+, \mathcal{V}_1^+, \ldots, \mathcal{V}_r^+, \mathcal{V}_\infty^+)$ is a partition of $\mathcal{V}^+$ (same applies to $\mathcal{V}^-$ and $\overline{\mathcal{E}}$). The component $\mathcal{V}_0$ is called the *minimal inconsistent part*, the component $\mathcal{V}_\infty$ is called the *maximal inconsistent part*, and the remaining components are called the *consistent parts*. In terms of those components, the following properties arise.

Definition 2.26 (Biadjacency matrix). Let $\overline{\mathcal{G}} = (\mathcal{V}^+ \cup \mathcal{V}^-, \overline{\mathcal{E}})$ be the bipartite graph defined in Definition 2.25. The *biadjacency matrix* of $\overline{\mathcal{G}}$ is the matrix $\Gamma \in \{0,1\}^{n \times (n+m)}$ with entries $\Gamma_{ji} = \begin{cases} 1, & \text{if } (v_i^+, v_j^-) \in \overline{\mathcal{E}}, \\ 0, & \text{otherwise.} \end{cases}$

Proposition 2.8 (From [25, 26]). *Let $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ be a bipartite graph and consider its DM-components. We have the following properties:*

1. *A maximum matching on $\overline{\mathcal{G}}$ is a union of maximum matchings on the DM-components $\overline{\mathcal{G}}_i$, $i \in \{0, 1, \ldots, r, \infty\}$;*
2. *A node v_i^- in $\mathcal{V}_0^-$ (equiv. in $\mathcal{V}_\infty^+$, or in $\mathcal{V}_i^+$ or $\mathcal{V}_i^-$, $i \in \{1, \ldots, r\}$) is covered by* any *maximum matching on $\overline{\mathcal{G}}$;*
3. *A node $v_i^+ \in \mathcal{V}^+$ belongs to the minimal inconsistent part $\mathcal{V}_0^+$ if and only if there exists a maximum matching on $\overline{\mathcal{G}}$ that does not cover v_i^+;*
4. *A node $v_i^- \in \mathcal{V}^-$ belongs to the maximal inconsistent part $\mathcal{V}_\infty^-$ if and only if there exists a maximum matching on $\overline{\mathcal{G}}$ that does not cover v_i^-;*
5. *There exist permutation matrices $U \in \{0,1\}^{n \times n}$ and $V \in \{0,1\}^{(n+m) \times (n+m)}$*

s.t. $U^\top \Gamma V$ *assumes the upper-block triangular form*



with overconstrained block c_0 *corresponding to the minimal inconsistent part* $\mathcal{V}_0$*, well-constrained blocks corresponding to the DM-components* $\mathcal{V}_i$*,* $i \in \{1, \ldots, r\}$*, and underconstrained block* c_∞ *corresponding to the maximal inconsistent part* $\mathcal{V}_\infty$*.*

Remark 2.20. Given a structured pair (A, B), the maximal inconsistent part $\mathcal{V}_\infty$, associated with the underconstrained block c_∞, corresponds, under the assumptions that will be stated in Proposition 2.9, to those nodes of $\mathcal{G}$ whose action excites multiple coordinates of the columns of the controllability matrix K and cannot be separated by elementary operations. These nodes typically represent dilations. Consequently, they do not belong to the fixed controllable subspace, since the nm columns of K are not structurally sufficient to isolate the individual state components they influence.

Lemma 2.2 (From [27]). *Given a structured irreducible pair* (A, B)*, its associated bipartite graph* $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ *and its DM-components, then* $|[A\ B\ e_i]|_g = |[A\ B]|_g + 1$ *if and only if* v_i^- *belongs to the set* $\mathcal{V}_\infty^-$ *of the DM-decomposition of* $\overline{\mathcal{G}}$*; otherwise,* $|[A\ B\ e_i]|_g = |[A\ B]|_g$*.*

Remark 2.21. As proven in Proposition 7 of [27], the generic rank defect in $[A\ B]$ can be quantified from the DM-decomposition of $\overline{\mathcal{G}}$ and corresponds to $|\mathcal{V}_\infty^-| - |\mathcal{V}_\infty^+|$. In particular, $|[A\ B]|_g = n$ if and only if $\mathcal{V}_\infty = \emptyset$, which corresponds to the existence of a perfect matching in $\overline{\mathcal{G}}$ from Proposition 2.7.

Remark 2.22. Finding the DM-decomposition of a bipartite graph requires first obtaining a maximum matching in the graph and, in a second step, using alternating path techniques. Since the alternating path techniques are of lower complexity, the complexity of obtaining the DM-decomposition is just as the complexity of maximum matching algorithm [28].

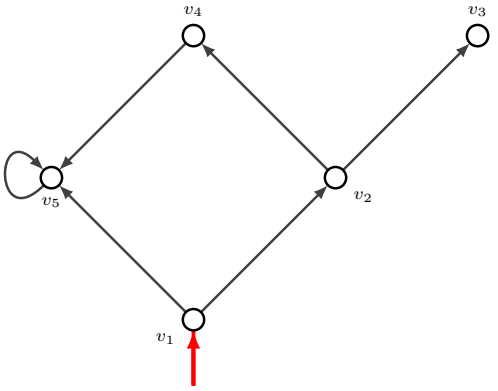
With Lemma 2.2 it is possible to identify only a subset of the nodes that do not belong to $\mathcal{R}^g$, allowing only a partial characterization of $\mathcal{R}^g$. The full characterization of $\mathcal{R}^g$ is given in the next Proposition.

Proposition 2.9 (Proposition 4 of [13]). *Given a structured irreducible pair*

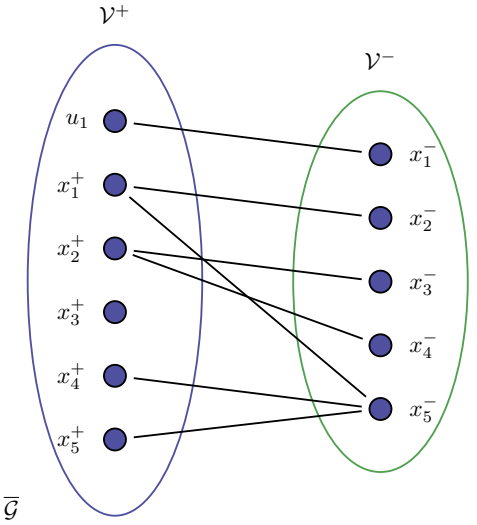
(A, B), its associated bipartite graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ and its DM-components, assume that $d = |[A\ B]|_g$. Then $v_i \in \mathcal{R}^g$ if and only if $v_i^- \notin \mathcal{V}_\infty^-$.

Example 2.5

Consider the following structured network $\mathcal{G}$ with $\mathcal{B} = \{v_1\}$ associated with the structured pair (A, B),



from which we can associate the bipartite graph $\overline{\mathcal{G}}$ given below.



We note by applying Theorem 2.5 to $\mathcal{G}$ that $d = 4$, and by applying Proposition 2.7 to $\overline{\mathcal{G}}$ that there exists a maximum matching of cardinality 4. Therefore, $|[A\ B]|_g = 4 = d$ and we can apply Proposition 2.9 to fully characterize $\mathcal{R}^g$. In fact, it is possible to compute the DM-components of $\overline{\mathcal{G}}$, in agreement with Proposition 2.8. The DM-decomposition of $\overline{\mathcal{G}}$ is shown in Figure 2.5. We note that $x_3^-, x_4^- \in \mathcal{V}_\infty^-$, thus $\mathcal{R}^g = \{v_1, v_2, v_5\}$. As a side remark, we note that v_5 with its self-loop constitutes a zero out-degree cycle, hence it must be in $\mathcal{R}^g$ by Proposition 2.6.

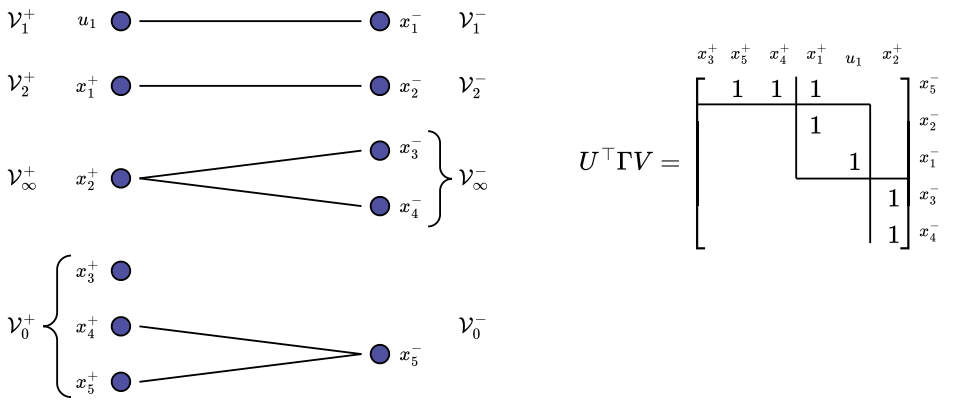
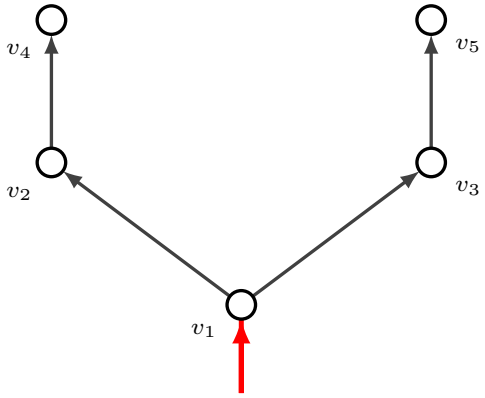


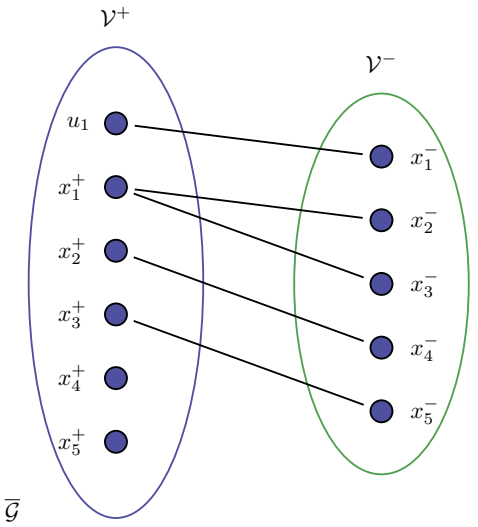
Figure 2.5: *Dulmage-Mendelsohn decomposition of* $\overline{\mathcal{G}}$.

Example 2.6

Consider the following structured network $\mathcal{G}$ with $\mathcal{B} = \{v_1\}$ associated with the structured pair (A, B),



from which we can associate the bipartite graph $\overline{\mathcal{G}}$ given below.



We note by applying Theorem 2.5 to $\mathcal{G}$ that $d = 3$, and by applying Proposition 2.7 to $\overline{\mathcal{G}}$, that there exists a maximum matching of cardinality 4. Therefore, $|[A\ B]|_g > d$ and we cannot apply Proposition 2.9 to fully characterize $\mathcal{R}^g$. In fact, the DM-decomposition of $\overline{\mathcal{G}}$ would give $x_2^-, x_3^- \in \mathcal{V}_\infty^-$. However, we can easily verify that the controllable subspaces associated with the structural pairs $(A, [B\ e_4])$ and $(A, [B\ e_5])$ have dimension $4 > 3$, therefore $\mathcal{R}^g = \{v_1\}$ only. On the other hand, nodes v_2 and v_3 are critical for controllability. In fact, we note that A is nilpotent of order 3, thus there exists a largest Jordan block of size 3 corresponding to the zero eigenvalue. We note that the nodes of $\mathcal{G}$ have in-degree ≤ 1, thus from [19, 20] there exists a proper partition of $\mathcal{G}$, namely the chains $\{v_1, v_2, v_4\}$ and $\{v_3, v_5\}$ or $\{v_1, v_3, v_5\}$ and $\{v_2, v_4\}$ constitute two admissible partitions for $\mathcal{G}$. The Jordan form of A is therefore $J_3(0) \oplus J_2(0)$. The control input in v_1 represents a generalized eigenvector of index 3, in fact, assuming $\mathcal{G}$ unweighted for simplicity, $Ae_1 = e_2 + e_3$, $A(e_2 + e_3) = e_4 + e_5$, and $A(e_4 + e_5) = 0$. If an additional control input is placed in v_2 or v_3, the remaining chain in the partition will be controlled. In fact, e_3 is a generalized eigenvector of index 2 s.t. $Ae_3 = e_5$ and $Ae_5 = 0$ (a similar reasoning applies to v_2). We obtain the transformation by stacking the generalized eigenvectors as $T = [e_4 + e_5,\ e_2 + e_3,\ e_1,\ e_5,\ e_3]$, so that $T^{-1}AT = J_3(0) \oplus J_2(0)$. Therefore, taking $\overline{B} = [B\ e_3]$ gives $T^{-1}\overline{B} = [0\ 0\ 1\ 0\ 1]^\top$, yielding a controllable pair $(A, \overline{B})$, in agreement with Remark 2.5.

We conclude this section with some comments on the maximal A-invariant output-nulling node set and minimal A-invariant input-containing node set. We note that whenever $\mathcal{R}^\circ \subset \mathcal{V}$ or $\mathcal{Q}^\circ \neq \emptyset$, controllability or observability cannot be achieved. However, $\mathcal{R}^\circ = \mathcal{V}$ or $\mathcal{Q}^\circ = \emptyset$ are equivalent to the controllability or observability of the structured pairs (A, B) or (C, A) only if there exist a perfect matching in the bipartite graphs associated with such pairs. In fact, as for the controllability, it is easy to show by duality that if $\mathcal{Q}^\circ = \emptyset$, structural observability is achieved

when $\left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\|_g = n$, i.e., when the transposed graph associated with the structured pair (C, A) is spanned by a disjoint union of cycles and stems rooted in $\mathcal{C}$.

2.3 Controlled and conditioned invariants

In this section, the notion of A-invariant subspace is extended to *controlled A-invariant subspace* (also called (A, B)-invariant subspace) and to *conditioned A-invariant subspace* (also called (C, A)-invariant subspace). Those notions have been introduced independently by Basile and Marro [10, 11], and by Wonham [29, 30]. Similarly to the previous section, a characterization of controlled/conditioned A-invariance can be made in network terms, in particular the corresponding notions of maximal input-containing and minimal output-nulling A-invariant subspaces can be extended to the notions of minimal input-containing conditioned A-invariant subspaces and maximal output-nulling controlled A-invariant subspaces, and their associated minimal input-containing conditioned A-invariant node sets and maximal output-nulling controlled A-invariant node sets.

It is in fact possible to associate with these structures, defined for vector subspaces, their analogous representations in terms of sets of nodes. These sets inherit the same limitations discussed in the previous section. Nevertheless, when applied to problems such as disturbance decoupling in large-scale networks, their intuitive and simple characterization leads to remarkable practical solutions.

Subspace-based characterization

Definition 2.27 (Controlled A-invariant subspace). Given a pair (A, B), a subspace $\mathcal{Z} \subseteq \mathbb{R}^n$ is called a controlled A-invariant (or an (A, B)-invariant) if $A\mathcal{Z} \subseteq \mathcal{Z} + \operatorname{Im} B$.

Definition 2.28 (Conditioned A-invariant subspace). Given a pair (C, A), a subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is called a conditioned A-invariant (or an (C, A)-invariant) if $A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$.

Definition 2.29 ((C, A, B)-pair). A pair of subspaces $(\mathcal{S}, \mathcal{Z})$ is called a (C, A, B)-pair if

- $\mathcal{S} \subseteq \mathcal{Z}$;
- $\mathcal{S}$ is (C, A)-invariant;
- $\mathcal{Z}$ is (A, B)-invariant.

Note that any A-invariant subspace is also (A, B)-invariant for any B and (C, A)-invariant for any C. The origin and the whole state-space satisfy these properties.

There exist equivalent characterizations of controlled/conditioned A-invariance, given in the following.

Proposition 2.10 (Theorem 4.2 of [31] and Property 4.1.3 of [10]). *$\mathcal{Z}$ is an (A, B)-invariant subspace if and only if any of the following equivalent conditions is met:*

(a) If Z is the basis matrix of $\mathcal{Z}$, $\exists$ matrices X, U s.t. $AZ = ZX + BU$;

(b) $\exists$ a matrix $F \in \mathbb{R}^{m\times n}$, called a "friend" of $\mathcal{Z}$, s.t. $\mathcal{Z}$ is $(A - BF)$-invariant, i.e., $(A - BF)\mathcal{Z} \subseteq \mathcal{Z}$.

Proposition 2.11 (Theorem 5.5 of [31]). *$\mathcal{S}$ is a (C, A)-invariant subspace if and only if $\exists$ a matrix $H \in \mathbb{R}^{n\times p}$, called a "friend" of $\mathcal{S}$, s.t. $\mathcal{S}$ is $(A - HC)$-invariant, i.e., $(A - HC)\mathcal{S} \subseteq \mathcal{S}$.*

Corollary 2.2. *$\mathcal{W}$ is an (A, B)-invariant and a (C, A)-invariant subspace if and only if $\exists$ a matrix $G \in \mathbb{R}^{m\times p}$, called a "friend" of $\mathcal{W}$, s.t. $\mathcal{W}$ is $(A - BGC)$-invariant, i.e., $(A - BGC)\mathcal{W} \subseteq \mathcal{W}$.*

The definitions of controlled/conditioned A-invariance can be further extended to output-nulling (A, B)-invariants and input-containing (C, A)-invariants. Since those objects are used for the disturbance decoupling problem, where the inputs to be contained are disturbances that need to be made invisible to the targets, we will call those extensions target-nulling (A, B)-invariants and disturbance-containing (C, A)-invariants. It is well known (see, e.g., [10]) that the set[6] of all the (C, A)-invariants containing a given subspace $\operatorname{Im} D$ and the set of all the (A, B)-invariants contained in a given subspace $\ker T$ admit a minimum and a maximum, respectively. Referring to the system (2.1), those are computed by the recursions given in the following Propositions.

Proposition 2.12 (Maximal target-nulling (A, B)-invariant subspace (Algorithm 4.1.2 of [10])). *The maximal target-nulling (A, B)-invariant subspace, denoted by $\mathcal{Z}^*$, is the maximal (A, B)-invariant subspace contained in $\ker T$, and can be computed in at most n steps, that is when $\mathcal{Z}_{k+1} = \mathcal{Z}_k =: \mathcal{Z}^*$ for $k \leq n - 1$,*

- $\mathcal{Z}_0 = \ker T$;
- $\mathcal{Z}_k = \ker T \cap A^{-1}(\mathcal{Z}_{k-1} + \operatorname{Im} B), \quad k \in \{1, \ldots, n - 1\}$.

Proposition 2.13 (Minimal disturbance-containing (C, A)-invariant subspace (Algorithm 4.1.1 of [10])). *The minimal disturbance-containing (C, A)-invariant subspace, denoted by $\mathcal{S}^*$, is the minimal (C, A)-invariant subspace con-*

[6] These structures are sets of subspaces under the operations of intersection, sum and set-inclusion, encoding their algebraic and geometric relationships. Their formal definition is given in Section 2.3.1.

taining $\operatorname{Im} D$*, and can be computed in at most* n *steps, that is when* $\mathcal{S}_{k+1} = \mathcal{S}_k =: \mathcal{S}^*$ *for* $k \leq n-1$*,*

- $\mathcal{S}_0 = \operatorname{Im} D$*;*
- $\mathcal{S}_k = \operatorname{Im} D + A(\mathcal{S}_{k-1} \cap \ker C), \quad k \in \{1, \dots, n-1\}$.

In the following, we extend the notions of controlled and conditioned A-invariance to sets of nodes.

Node set-based characterization

Definition 2.30 (Controlled A-invariant set of nodes). Given a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ and a matrix B of elementary columns associated to the set of inputs $\mathcal{B} \subseteq \mathcal{V}$, a subset of the network nodes $\mathcal{Z} \subseteq \mathcal{V}$ is an (A, B)-invariant set of nodes if $\forall\, (v_i,\, v_j) \in \mathcal{E}, v_i \in \mathcal{Z} \implies v_j \in \mathcal{Z} \cup \mathcal{B}$.

Definition 2.31 (Conditioned A-invariant set of nodes). Given a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ and a matrix C of elementary rows associated to the set of measurements $\mathcal{C} \subseteq \mathcal{V}$, a subset of the network nodes $\mathcal{S} \subseteq \mathcal{V}$ is a (C, A)-invariant set of nodes if $\forall\, (v_i,\, v_j) \in \mathcal{E}, v_i \in \mathcal{S}, v_i \notin \mathcal{C} \implies v_j \in \mathcal{S}$.

Remark 2.23. By means of the identification $e_i \sim v_i$, the controlled/conditioned node sets are trivially also controlled/conditioned A-invariant subspaces, therefore for controlled/conditioned node sets the conditions of Propositions 2.10 and 2.11 still hold true. Moreover, using Definitions 2.30 and 2.31, similar definitions in terms of node sets can be made for a set of nodes both controlled and conditioned A-invariant, denoted by $\mathcal{W}^\circ$, and for a (C, A, B)-pair of node sets.

It is possible to extend the notions of controlled/conditioned node sets to target-nulling and disturbance-containing controlled/conditioned A-invariants node sets. In the set of all the target-nulling (A, B)-invariant node sets, there exist a maximum, given by the union of all such node sets. Similarly, in the set of all the disturbance-containing (C, A)-invariant node sets, there exist a minimum, given by the intersection of all such node sets. Those sets are the corresponding node sets of the subspaces $\mathcal{Z}^*$ and $\mathcal{S}^*$ of Propositions 2.12 and 2.13, and are computed in the following Propositions, inspired by the recursions given in [32].

Proposition 2.14 (Maximal target-nulling (A, B)-invariant node set). *Given the subsets of nodes* $\mathcal{B}$ *and* $\mathcal{T}$*, the maximal* (A, B)*-invariant set of nodes* $\mathcal{Z}^\circ$ *contained in* $\mathcal{V} \setminus \mathcal{T}$ *can be computed in at most* n *steps, that is when* $\mathcal{Z}_{k+1} = \mathcal{Z}_k =: \mathcal{Z}^\circ$ *for* $k \leq n-1$*,*

- $\mathcal{Z}_0 = \mathcal{V} \setminus \mathcal{T}$*;*

- $\mathcal{Z}_k = \mathcal{Z}_{k-1} \setminus \{v_i \in \mathcal{Z}_{k-1} \text{ s.t. } \exists \textit{ at least one } (v_i, v_j) \in \mathcal{E} \text{ s.t. } v_j \notin \mathcal{Z}_{k-1} \cup \mathcal{B}\}$, $k \in \{1, \dots, n-1\}$.

Proof: See Remark 2.24. ∎

Proposition 2.15 (Minimal disturbance-containing (C, A)-invariant node set). *Given the subsets of nodes $\mathcal{C}$ and $\mathcal{D}$, the minimal (C, A)-invariant set of nodes $\mathcal{S}^\circ$ containing $\mathcal{D}$ can be computed in at most n steps, that is when $\mathcal{S}_{k+1} = \mathcal{S}_k =: \mathcal{S}^\circ$ for $k \leq n-1$,*

- $\mathcal{S}_0 = \mathcal{D}$;
- $\mathcal{S}_k = \mathcal{S}_{k-1} \cup \{v_j \notin \mathcal{S}_{k-1} \text{ s.t. } \exists \textit{ at least one } (v_i, v_j) \in \mathcal{E} \text{ s.t. } v_i \in \mathcal{S}_{k-1} \setminus \mathcal{C}\}$, $k \in \{1, \dots, n-1\}$.

Proof: $\mathcal{D} \subseteq \mathcal{S}^\circ$ by construction. Assume by contradiction that $\mathcal{S}^\circ$ is not (C, A)-invariant, then there must exist at least one node v_i in $\mathcal{S}^\circ \setminus \mathcal{C}$, a node v_j in $\mathcal{S}^{\circ\perp}$ and an edge (v_i, v_j). This contradicts the definition of conditioned A-invariant node set, given in Definition 2.31. Minimality is trivial by the construction of the scheme, while the convergence in at most n steps is guaranteed by the fact that the sequence is non-decreasing and the total number of nodes is bounded by n. ∎

The following duality result is derived in [10] and in Proposition 7 of Paper B.

Proposition 2.16 (Duality - $\mathcal{S}^*, \mathcal{Z}^*$ and $\mathcal{S}^\circ, \mathcal{Z}^\circ$). *Denoting by $\mathcal{S}^* = \mathcal{S}^*(C, A; \operatorname{Im} D)$ and $\mathcal{S}^\circ = \mathcal{S}^\circ(\mathcal{C}, A; \mathcal{D})$ the minimal disturbance-containing (C, A)-invariant subspace and node set, respectively, and by $\mathcal{Z}^* = \mathcal{Z}^*(A, B; \ker T)$ and $\mathcal{Z}^\circ = \mathcal{Z}^\circ(A, \mathcal{B}; \mathcal{V} \setminus \mathcal{T})$ the maximal target-nulling (A, B)-invariant subspace and node set, respectively, the following dualities hold*

- $\mathcal{S}^{*\perp} = \mathcal{Z}^*(A^\top, C^\top; \ker D^\top), \quad \mathcal{Z}^{*\perp} = \mathcal{S}^*(B^\top, A^\top; \operatorname{Im} T^\top)$;
- $\mathcal{S}^{\circ\perp} = \mathcal{Z}^\circ(A^\top, \mathcal{C}; \mathcal{V} \setminus \mathcal{D}), \quad \mathcal{Z}^{\circ\perp} = \mathcal{S}^\circ(\mathcal{B}, A^\top; \mathcal{T})$.

Remark 2.24. The proof of Proposition 2.14 follows by duality of the one of Proposition 2.15. Note also that the properties of containing the disturbances and of being invisible to the targets depend exclusively on the initializations $\mathcal{S}_0$, $\mathcal{Z}_0$.

An example of computation of $\mathcal{Z}^\circ$ and $\mathcal{S}^\circ$ is shown in Figures 2.6 and 2.7. On the same graph, in Figure 2.8 a configuration of control inputs and measurement nodes for which there exists a node set, denoted by $\mathcal{W}^\circ$, both controlled and conditioned A-invariant is shown. Likewise, in Figure 2.9 a configuration of control inputs and measurement nodes for which there exists a (C, A, B)-pair of node sets is shown.

We now discuss the difference between $\mathcal{Z}^*$, $\mathcal{S}^*$ and $\mathcal{Z}^\circ$, $\mathcal{S}^\circ$. In particular, the following Theorem states that the subspaces $\mathcal{Z}^*$ and $\mathcal{S}^*$ need not coincide with the node sets $\mathcal{Z}^\circ$ and $\mathcal{S}^\circ$.

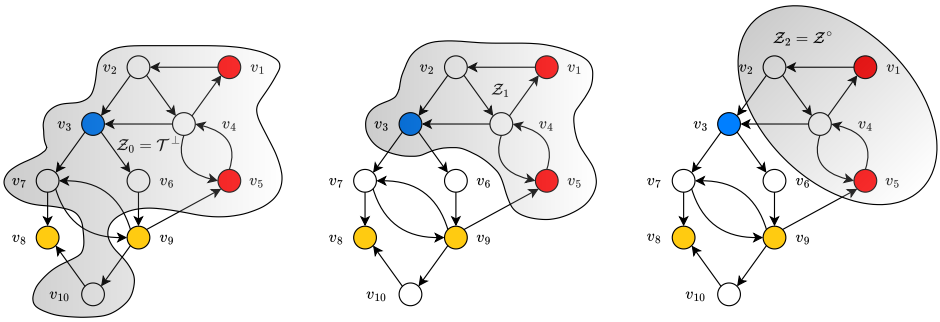


Figure 2.6: *Computation of $\mathcal{Z}^\circ$ using Proposition 2.14. Target nodes are in yellow, disturbances are in red, control inputs in blue.*

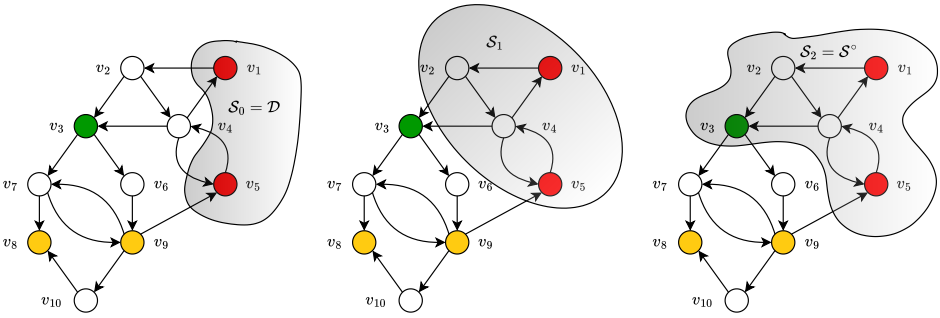


Figure 2.7: *Computation of $\mathcal{S}^\circ$ using Proposition 2.15. Target nodes are in yellow, disturbances are in red, measurements in green.*

Theorem 2.9. *As vector subspaces, $\mathcal{S}^\circ \supseteq \mathcal{S}^*$ and $\mathcal{Z}^\circ \subseteq \mathcal{Z}^*$. In particular,*

(a) If $x \in \mathcal{S}^*$ is s.t. $x_i \neq 0 \implies v_i \in \mathcal{S}^\circ$;

(b) If $v_i \in \mathcal{Z}^\circ \implies x \in \mathcal{Z}^*$, $x_i \neq 0$.

Proof: We note that proving items (a) and (b) of the Theorem is sufficient to prove the inclusions $\mathcal{S}^\circ \supseteq \mathcal{S}^*$ and $\mathcal{Z}^\circ \subseteq \mathcal{Z}^*$. We prove item (a) and note that (b) follows by duality. The proof of item (a) is done by contradiction and requires the reasoning and the tools developed in Paper B. By construction of $\mathcal{S}^\circ$, node $v_i \notin \mathcal{S}^\circ$ iff $\nexists\, \mathcal{P}^{\mathcal{D},v_i} \neq \emptyset$ or $\exists\, \mathcal{P}^{\mathcal{D},v_i} \neq \emptyset$ s.t. $v_i \notin \partial_-(\mathcal{S}^\circ, A)$ and $\mathcal{P}^{\mathcal{D},v_i} \cap \partial_-(\mathcal{S}^\circ, A) \neq \emptyset$ (note that this is a direct consequence of Lemma 3 of Paper B). Moreover, $\mathcal{D} \subseteq \mathcal{S}^\circ$ therefore $v_i \notin \mathcal{D}$. Now note that $\partial_-(\mathcal{S}^\circ, A) \subseteq \mathcal{C}$ and that if $v_i \in \mathcal{C}$ then $x_i = 0$ since $e_i \notin \ker e_i^\top \supseteq \ker C$. If $\mathcal{P}^{\mathcal{D},v_i} = \emptyset$ then $\forall\, v_j \in \partial_-(\mathcal{S}^\circ, A)$, $x = A^\ell e_j$ has $x_i = 0$, $\forall \ell \leq n-1$. If $\mathcal{P}^{\mathcal{D},v_i} \neq \emptyset$ then $\forall\, v_j \in \partial_-(\mathcal{S}^\circ, A)$, $\exists\, \ell$ s.t. $x = A^\ell e_j$ has $x_i \neq 0$. However, $e_j \in \operatorname{Im} C^\top$ because of $\partial_-(\mathcal{S}^\circ, A) \subseteq \mathcal{C}$, thus $e_j \notin \mathcal{S}_{\ell-1} \cap \ker C$ and such an x cannot be in $\mathcal{S}^*$. ∎

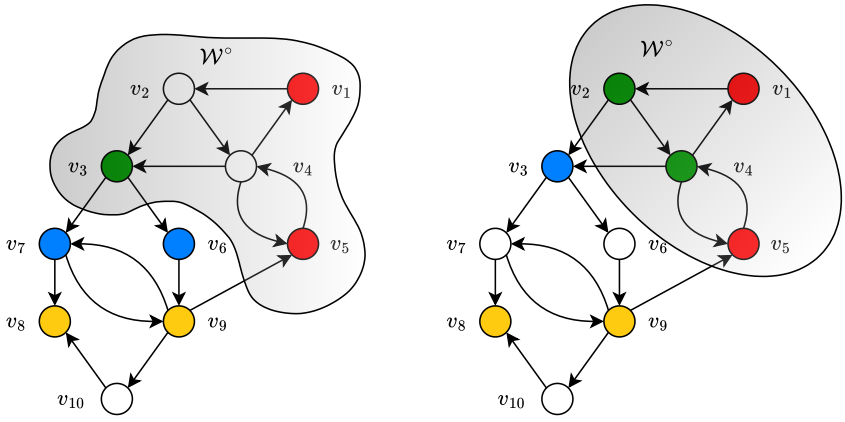


Figure 2.8: *Both configurations of $\mathcal{B}$ and $\mathcal{C}$ give a set of nodes $\mathcal{W}^\circ$ both controlled and conditioned A-invariant. Target nodes are in yellow, disturbances are in red, control inputs in blue and measurements are in green.*

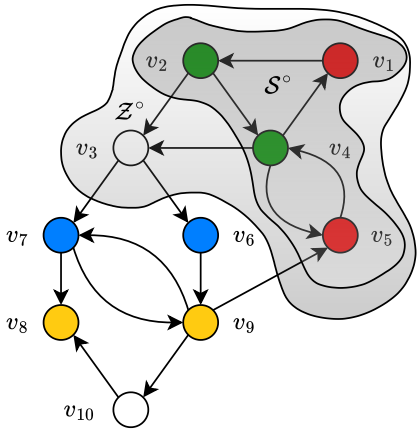


Figure 2.9: *The pair of node sets $(\mathcal{S}^\circ, \mathcal{Z}^\circ)$ is a (C, A, B)-pair. Target nodes are in yellow, disturbances are in red, control inputs in blue and measurements are in green.*

Remark 2.25. Theorem 2.9 shows that there are no coordinates in $\mathcal{S}^*$ that are not nodes of $\mathcal{S}^\circ$, and there are no nodes in $\mathcal{Z}^\circ$ that are not coordinates in $\mathcal{Z}^*$. In the example that follows it is shown that the nodes in $\mathcal{S}^\circ$ can exceed (but always include) the nonzero coordinates of a generic element of $\mathcal{S}^*$. The reverse implications of items (a) and (b) of Theorem 2.9 are not true in general.

The following example shows that not all the coordinates touched in $\mathcal{S}^*$ appear in $\mathcal{S}^\circ$ (which was the case for $\mathcal{R}$ and $\mathcal{R}^\circ$), since the intersection with $\ker C$ annihilates some nodes in $\mathcal{S}^\circ$.

Example 2.7

Consider the network depicted in Figure 2.10, with $\mathcal{D} = \{v_3\}$, $\mathcal{T} = \{v_5\}$, $\mathcal{C} = \{v_4\}$ and $\mathcal{B} = \{v_2\}$, and take a generic matrix $A = (a_{ij}) \in \mathcal{A}$.

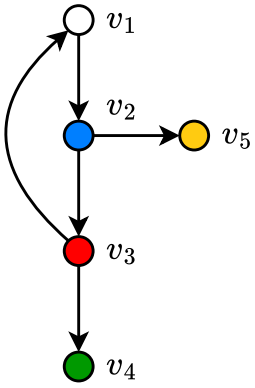


Figure 2.10: *Disturbances are in red, targets in yellow, control inputs in blue, whereas the measurement nodes are in green.*

By means of Propositions 2.14 and 2.15 we get $\mathcal{S}^\circ = \mathcal{V}$ and $\mathcal{Z}^\circ = \{v_1, v_3, v_4\}$. Now we compute $\mathcal{S}^*$ by means of the classical recursion in Proposition 2.13. We note that $D = e_3$, $C = e_4^\top$, thus $\operatorname{Im} D = \mathcal{S}_0 = \mathcal{S}_0 \cap \ker C = \operatorname{span}\{e_3\}$. Since $Ae_3 = [a_{13}\ 0\ 0\ a_{43}\ 0]^\top$ it follows that $\mathcal{S}_1 = \operatorname{span}\{e_3\} + A(\mathcal{S}_0 \cap \ker C) = \operatorname{span}\{e_3, [a_{13}\ 0\ 0\ a_{43}\ 0]^\top\}$. Now note that also $\mathcal{S}_1 \cap \ker C = \operatorname{span}\{e_3\} = \mathcal{S}_0 \cap \ker C$. In fact, $x \in \mathcal{S}_1$ iff $x = [\alpha a_{13}\ 0\ \beta\ \alpha a_{43}\ 0]^\top$, $\alpha, \beta \in \mathbb{R}$ and $y \in \ker C$ iff $y = [\gamma\ \delta\ \lambda\ 0\ \xi]^\top$, $\gamma, \delta, \lambda, \xi \in \mathbb{R}$, thus x and y belong to the intersection iff $\lambda = \beta$ and $\alpha = \delta = \xi = 0$, i.e., iff $x, y \in \operatorname{span}\{e_3\}$. We therefore conclude that $\mathcal{S}_2 = \mathcal{S}_1 =: \mathcal{S}^*$. From $\mathcal{S}^* = \operatorname{span}\{e_3, [a_{13}\ 0\ 0\ a_{43}\ 0]^\top\}$ we note that not all the nodes in $\mathcal{S}^\circ$ appear as nonzero coordinates in the generators of $\mathcal{S}^*$, but that all the nonzero coordinates that appear in $\mathcal{S}^*$ are also nodes of $\mathcal{S}^\circ$, as proven in Theorem 2.9. We also note that, as for the fixed controllable subspace $\mathcal{R}^g$, node v_3 is fixed, meaning that $\mathcal{S}^g = \operatorname{span}\{e_3\}$ is in $\mathcal{S}^*$ independently of the entries of A (also note that it is not (C, A)-invariant). We conclude this example by showing that, for this case, $\mathcal{Z}^* = \mathcal{Z}^\circ$. In fact, by applying the recursion of Proposition 2.12, we note that since $B = e_2$ and $T = e_5^\top$ we get $\mathcal{Z}_0 = \ker T = \operatorname{span}\{e_1, e_2, e_3, e_4\} = \mathcal{Z}_0 + \operatorname{Im} B$. Now taking $x \in \mathbb{R}^5$ and imposing $Ax = [a_{13}x_3\ a_{21}x_1\ a_{32}x_2\ a_{43}x_3\ a_{52}x_2]^\top \in \mathcal{Z}_0 + \operatorname{Im} B$ means $x_2 = 0$, since $a_{52} \neq 0$. Thus $A^{-1}(\mathcal{Z}_0 + \operatorname{Im} B) = \ker e_2^\top$, so $\mathcal{Z}_1 = \ker T \cap \ker e_2^\top = \ker[e_5\ e_2]^\top = \operatorname{span}\{e_1, e_3, e_4\}$. By noting that $\mathcal{Z}_1 + \operatorname{Im} B = \mathcal{Z}_0$, we conclude that $\mathcal{Z}_2 = \mathcal{Z}_1 =: \mathcal{Z}^*$.

Example 2.7, and in particular Theorem 2.9 and Remark 2.25, highlight some limitations of solving the disturbance decoupling problem (both by state, output and dynamical feedback) by means of controlled and conditioned invariant node sets. In fact, while the equivalent conditions on $\mathcal{Z}^*$ and $\mathcal{S}^*$ are necessary and

sufficient for the solvability of the DDP, the conditions written in terms of the sets $\mathcal{Z}^\circ$ and $\mathcal{S}^\circ$ are only sufficient.

Examples where the three problems are solvable by using the node sets approach are depicted in Figure 2.6 for the DDPSF, in Figure 2.8 for the DDPOF and in Figure 2.9 for the DDPDF. Theorem 2.9 is crucial, because the non-equivalence of subspaces and node sets implies that the node set conditions we obtain in Paper B below are only sufficient but not necessary (unlike the corresponding vector subspace conditions). In particular, if $\mathcal{W}$ denotes a both controlled and conditioned A-invariant subspace and $\mathcal{W}^\circ$ a both controlled and conditioned A-invariant node set, we have that:

1. $\mathcal{D} \subseteq \mathcal{Z}^\circ \implies \operatorname{Im} D \subseteq \mathcal{Z}^* \iff$ **DDPSF** is solvable [30];

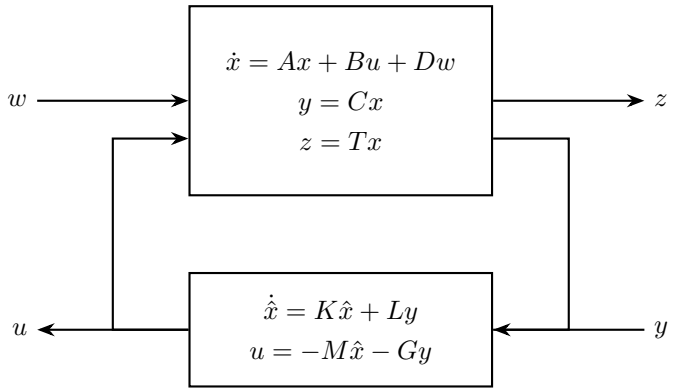
2. $\mathcal{D} \subseteq \mathcal{W}^\circ \subseteq \mathcal{V} \setminus \mathcal{T} \implies \operatorname{Im} D \subseteq \mathcal{W} \subseteq \ker T \iff$ **DDPOF** is solvable [33];

3. $\mathcal{S}^\circ \subseteq \mathcal{Z}^\circ \implies \mathcal{S}^* \subseteq \mathcal{Z}^* \iff$ **DDPDF** is solvable [34].

This is clarified in the following example.

Example 2.8

Consider again the network depicted in Figure 2.10, with $\mathcal{D} = \{v_3\}$, $\mathcal{T} = \{v_5\}$, $\mathcal{C} = \{v_4\}$ and $\mathcal{B} = \{v_2\}$, and take a generic matrix $A = (a_{ij}) \in \mathcal{A}$. We computed the node sets $\mathcal{S}^\circ = \mathcal{V}$ and $\mathcal{Z}^\circ = \{v_1, v_3, v_4\}$, and the subspaces $\mathcal{S}^* = \operatorname{span}\{e_3, [a_{13}\ 0\ 0\ a_{43}\ 0]^\top\}$, and $\mathcal{Z}^* = \operatorname{span}\{e_1, e_3, e_4\}$. $\mathcal{D} \subseteq \mathcal{Z}^\circ$, and therefore the DDPSF is solvable. Now note that $\nexists \mathcal{W}^\circ$ both controlled and conditioned A-invariant set of nodes, hence the DDPOF based on the sets of nodes is not solvable. In fact, as proven in Theorem 2 of Paper B, in all the $\mathcal{D}$-to-$\mathcal{T}$ paths (e.g., $\{v_3, v_1, v_2, v_5\}$) there should be at least a sub-path of length 1 from $\mathcal{C}$ to $\mathcal{B}$. However, the DDPOF based on the subspaces is solvable in this case. We note that the subspace $\mathcal{S}^* = \operatorname{span}\{e_3, [a_{13}\ 0\ 0\ a_{43}\ 0]^\top\}$ is also controlled A-invariant and is contained in $\ker T$. In fact, $Ae_3 = a_{13}e_1 + a_{43}e_4 \in \mathcal{S}^*$ and $A[a_{13}\ 0\ 0\ a_{43}\ 0]^\top = a_{21}a_{13}e_2 \in \operatorname{Im} B$. The DDPOF is thus solvable for $\mathcal{W} = \mathcal{S}^*$ with the friend $G = \frac{a_{13}a_{21}}{a_{43}}$. Moreover, we note that $(\mathcal{S}^\circ, \mathcal{Z}^\circ)$ is not a (C, A, B)-pair while $(\mathcal{S}^*, \mathcal{Z}^*)$ is. Therefore the DDPDF is solvable using the (C, A, B)-pair $(\mathcal{S}^*, \mathcal{Z}^*)$, and it is possible to compute a dynamic compensator (see Appendix of Paper B for more details). In fact $G = \frac{a_{13}a_{21}}{a_{43}}$ is s.t. $(A - BGC)\mathcal{S}^* \subseteq \mathcal{Z}^*$, $F = [a_{21}\ 0\ 0\ 0\ 0]$ is a friend of $\mathcal{Z}^*$ and $H = [0\ \frac{a_{13}a_{21}}{a_{43}}\ 0\ 0\ 0]^\top$ is a friend of $\mathcal{S}^*$. From Theorem 13 of Paper B, the (full-order) compensator (K, L, M, G) can be derived for the closed-loop system of the following block diagram,



whose matrices A_c, D_c and T_c are defined in the system (8) in the Appendix of Paper B. With the dynamical feedback (K, L, M, G) there exists an A_c-invariant subspace $\mathcal{W}_c = \{\begin{bmatrix} x_{\mathcal{S}^*}^\top + x_{\mathcal{Z}^*}^\top & x_{\mathcal{Z}^*}^\top \end{bmatrix}^\top$ s.t. $x_{\mathcal{S}^*} \in \mathcal{S}^*, x_{\mathcal{Z}^*} \in \mathcal{Z}^*\}$ s.t. $\mathrm{Im} D_c \subseteq \mathcal{W}_c \subseteq \ker T_c$.

2.3.1 The subspace algebra framework

In the previous sections, we defined the notions of A-invariant, (A, B)-invariant, and (C, A)-invariant subspaces. In this section, we extend the theory to consider the structures that contain all such subspaces, summarize the basic properties and fundamental identities that arise when combining two or more of them, and discuss whether these structures are closed under the sum and intersection of their elements. The following definitions can be found in Appendix A.1.2 of [10] and in [35].

Definition 2.32 (Partially ordered set). A partially ordered set (or poset) is a pair $(P, \leq)$, where P is a set and $\leq$ is a binary relation on P that is

- *Reflexive*: $x \leq x$ for all $x \in P$;
- *Antisymmetric*: if $x \leq y$ and $y \leq x$, then $x = y$;
- *Transitive*: if $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition 2.33 (Lattice). A lattice $(L, \leq; \wedge, \vee)$ is a partially ordered set $(L, \leq)$ such that for every pair $x, y \in L$ there exist:

- A greatest lower bound of $\{x, y\}$ with respect to $\leq$, denoted by $x \wedge y$ and called the *meet* of x and y;
- A least upper bound of $\{x, y\}$ with respect to $\leq$, denoted by $x \vee y$ and called the *join* of x and y.

Remark 2.26. Take as an example a lattice of subspaces $(L, \subseteq; \cap, +)$, ordered by subspace inclusion, with join given by the sum of subspaces and meet given by their intersection. By definition of lattice, both the minimum element of L and the maximum element of L are obtained, respectively, by the intersection and the sum of all its elements.

Definition 2.34 (Modular lattice). A lattice $(L, \leq; \wedge, \vee)$ is called modular if for all $x, y, z \in L$ with $x \leq z$, $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 2.35 (Distributive lattice). A lattice $(L, \leq; \wedge, \vee)$ is said to be distributive if for all $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, or equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. A lattice that is not distributive is called non-distributive.

Definition 2.36 (Semilattice). A lower semilattice (also called meet-semilattice) is a partially ordered set $(S, \leq)$ such that $x \wedge y$ exists for every $x, y \in S$. An upper semilattice (also called join-semilattice) is defined analogously by requiring the existence of $x \vee y$ for all $x, y \in S$.

Remark 2.27. Consider a lower semilattice $(S, \subseteq; \cap)$. By definition, its minimum element is given by the intersection of all its elements. However, since it is not necessarily closed under the sum operation, it may admit a maximum, but this maximum cannot, in general, be obtained as the sum of all its elements. A dual reasoning applies to an upper semilattice $(S, \subseteq; +)$.

The set of all vector subspaces of $\mathbb{R}^n$ of dimension k is called the Grassmannian $\mathbf{Gr}_k(\mathbb{R}^n)$ and it can be proved that it is a differentiable manifold.

Definition 2.37 (Full Grassmannian). The full Grassmannian of $\mathbb{R}^n$ is the disjoint union of all Grassmannians of fixed dimension, i.e.,

$$\mathbf{Gr}(\mathbb{R}^n) = \bigsqcup_{k=0}^{n} \mathbf{Gr}_k(\mathbb{R}^n) = \{\mathcal{W} \subseteq \mathbb{R}^n \text{ s.t. } \mathcal{W} \text{ is a vector subspace}\}.$$

$\mathbf{Gr}(\mathbb{R}^n)$ is stratified by subspace dimension, where each stratum is a differential manifold. It can be verified that the elements of $\mathbf{Gr}(\mathbb{R}^n)$ form a lattice when ordered by subspace inclusion, with join given by the sum of subspaces and meet given by their intersection. Moreover, this lattice is modular and non-distributive.

Definition 2.38 (Grassmannian lattice). $(\mathbf{Gr}(\mathbb{R}^n), \subseteq; \cap, +)$ is a non-distributive modular lattice, with minimum $\{0\}$ and maximum $\mathbb{R}^n$ and s.t. $\forall\, \mathcal{W}_1, \mathcal{W}_2 \in \mathbf{Gr}(\mathbb{R}^n)$

- $\mathcal{W}_1 + \mathcal{W}_2$ is the smallest subspace containing $\mathcal{W}_1$ and $\mathcal{W}_2$;
- $\mathcal{W}_1 \cap \mathcal{W}_2$ is the largest subspace contained in both $\mathcal{W}_1$ and $\mathcal{W}_2$.

A graphical representation of the order structure in $(\mathbf{Gr}(\mathbb{R}^n), \subseteq; \cap, +)$ can be made by means of an *Hasse diagram*, and is depicted in Figure 2.11.

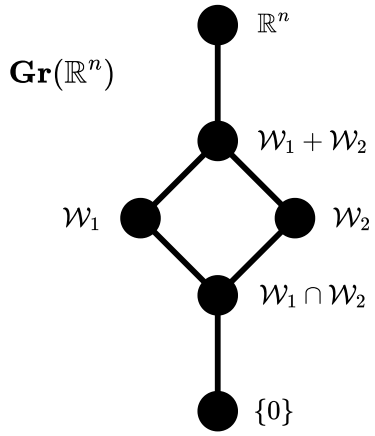


Figure 2.11: *Hasse diagram of* $(\mathbf{Gr}(\mathbb{R}^n), \subseteq; \cap, +)$.

Given a pair (A, B), the set of all input-containing A-invariant subspaces of $\mathbb{R}^n$ is defined as

$$\mathcal{J}(A, B) = \{\mathcal{W} \in \mathbf{Gr}(\mathbb{R}^n) \text{ s.t. } A\mathcal{W} \subseteq \mathcal{W} \text{ and } \operatorname{Im} B \subseteq \mathcal{W}\}.$$

It is possible to prove that $\mathcal{J}(A, B)$ is closed under the sum and the intersection of its elements, thus it forms a lattice with respect to those operations. Moreover, as for $(\mathbf{Gr}(\mathbb{R}^n), \subseteq; \cap, +)$, it is modular and non-distributive.

Definition 2.39 (Input-containing A-invariant lattice). $(\mathcal{J}(A, B), \subseteq; \cap, +)$ is a non-distributive modular lattice, whose maximum is $\mathbb{R}^n$ and whose minimum is the controllable subspace $\mathcal{R} = \bigcap_{\mathcal{W} \in \mathcal{J}(C,A)} \mathcal{W}$.

Similarly, given a pair (C, A), the set of all output-nulling A-invariant subspaces of $\mathbb{R}^n$ is defined as

$$\mathcal{J}(C, A) = \{\mathcal{W} \in \mathbf{Gr}(\mathbb{R}^n) \text{ s.t. } A\mathcal{W} \subseteq \mathcal{W} \text{ and } \mathcal{W} \subseteq \ker C\}.$$

Similar to $\mathcal{J}(A, B)$, it can be shown that also $\mathcal{J}(C, A)$ is closed under the sum and the intersection of its elements, thus it forms a lattice with respect to those operations. Moreover, as for $(\mathbf{Gr}(\mathbb{R}^n), \subseteq; \cap, +)$, it is modular and non-distributive.

Definition 2.40 (Output-nulling A-invariant lattice). $(\mathcal{J}(C, A), \subseteq; \cap, +)$ is a non-distributive modular lattice, whose minimum is $\{0\}$ and whose maximum is the unobservable subspace $\mathcal{Q} = \sum_{\mathcal{W} \in \mathcal{J}(C,A)} \mathcal{W}$.

Given a pair (C, A) and the subspace $\operatorname{Im} D$, the set of all disturbance-containing

(C, A)-invariant subspaces of $\mathbb{R}^n$ is defined as

$$\mathcal{J}(C, A; \operatorname{Im} D) = \{\mathcal{W} \in \mathbf{Gr}(\mathbb{R}^n) \text{ s.t. } A(\mathcal{W} \cap \ker C) \subseteq \mathcal{W} \text{ and } \operatorname{Im} D \subseteq \mathcal{W}\}.$$

It can be proven (see, e.g., Property 4.1.2 of [10]) that the intersection of any (C, A)-invariant subspace is again a (C, A)-invariant subspace, while the sum is not, in general. As a consequence, $\mathcal{J}(C, A; \operatorname{Im} D)$ is a lower semilattice with respect to the intersection operation and its minimum is obtained as the intersection of all its elements.

Definition 2.41 (Disturbance-containing (C, A)-invariant semilattice). $(\mathcal{J}(C, A; \operatorname{Im} D), \subseteq; \cap)$ is a non-distributive modular lower semilattice, whose maximum is $\mathbb{R}^n$ and whose minimum is the subspace $\mathcal{S}^* = \bigcap_{\mathcal{W} \in \mathcal{J}(C, A; \operatorname{Im} D)} \mathcal{W}$, defined in Proposition 2.13.

Similarly, given a pair (A, B) and the subspace $\ker T$, the set of all target-nulling (A, B)-invariant subspaces of $\mathbb{R}^n$ is defined as

$$\mathcal{J}(A, B; \ker T) = \{\mathcal{W} \in \mathbf{Gr}(\mathbb{R}^n) \text{ s.t. } A\mathcal{W} \subseteq \mathcal{W} + \operatorname{Im} B \text{ and } \mathcal{W} \subseteq \ker T\}.$$

It can be proven (see, e.g., Property 4.1.1 of [10]) that the sum of any (A, B)-invariant subspace is again an (A, B)-invariant subspace, while the intersection is not, in general. As a consequence, $\mathcal{J}(A, B; \ker T)$ is an upper semilattice with respect to the sum operation and its maximum is obtained as the sum of all its elements.

Definition 2.42 (Target-nulling (A, B)-invariant semilattice). $(\mathcal{J}(A, B; \ker T), \subseteq; +)$ is a non-distributive modular upper semilattice, whose minimum is $\{0\}$ and whose maximum is the subspace $\mathcal{Z}^* = \sum_{\mathcal{W} \in \mathcal{J}(A, B; \ker T)} \mathcal{W}$, defined in Proposition 2.12.

In the previous sections it has been shown that, when interpreted as vector subspaces, the following statements are true for the set of nodes defined in Propositions 2.3, 2.4, 2.14 and 2.15:

- $\mathcal{R}^\circ \in \mathcal{J}(A, B)$ and $\mathcal{R} \subseteq \mathcal{R}^\circ$;
- $\mathcal{Q}^\circ \in \mathcal{J}(C, A)$ and $\mathcal{Q}^\circ \subseteq \mathcal{Q}$;
- $\mathcal{S}^\circ \in \mathcal{J}(C, A; \operatorname{Im} D)$ and $\mathcal{S}^* \subseteq \mathcal{S}^\circ$;
- $\mathcal{Z}^\circ \in \mathcal{J}(A, B; \ker T)$ and $\mathcal{Z}^\circ \subseteq \mathcal{Z}^*$.

These node sets are not maximal or minimal elements in their respective lattice or semilattice structures; however, in the context of sets of nodes, they do represent the maximal or minimal node sets with such properties. The aforementioned relationships are exposed in the Hasse diagrams of Figure 2.12.

2.3.1.1 Additional considerations about the "fixed" subspaces

It has been shown that there exists a subspace $\mathcal{R}^g$ that generically belongs to $\mathcal{R}$, called the fixed controllable subspace. Such subspace is a node set and is contained

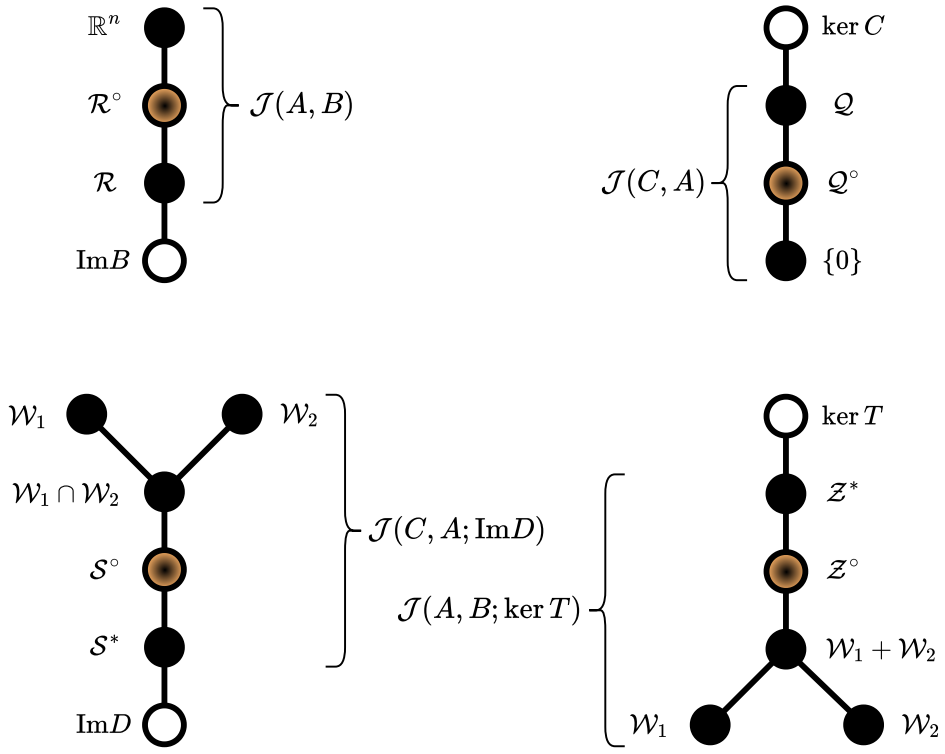


Figure 2.12: *Hasse diagrams of $\mathcal{J}(A,B)$, $\mathcal{J}(C,A)$, $\mathcal{J}(C,A;\mathrm{Im}\,D)$ and $\mathcal{J}(A,B;\ker T)$. The node sets have been partially colored in orange.*

in $\mathcal{R}$. It is possible to define the minimal set of nodes that contains $\mathcal{R}$, as the minimal subspace that generically contains $\mathcal{R}$, denoted by $\mathcal{R}_g$. Formally, a basis of $\mathcal{R}_g$ consists of the minimal number of canonical vectors s.t. a generic element of $\mathcal{R}$ can be written as a linear combination of those vectors. As a consequence, a basis of $\mathcal{R}_g$ consists of all and only the canonical basis vectors s.t. if a generic element of $\mathcal{R}$ has the i-th component that is nonzero then $e_i \in \mathcal{R}_g$. By construction, $\mathcal{R}_g$ is also a fixed subspace, in the sense that it is fixed with respect to the entries of A, and it is straightforward to prove that it coincides with $\mathcal{R}^\circ$. Moreover, $\mathcal{R}_g = \mathcal{R}^\circ$ is A-invariant, in fact it is the minimal A-invariant input-containing set of nodes. On the other hand, $\mathcal{R}^g$ is not in general A-invariant, since $\mathcal{R}$ is the minimal A-invariant subspace containing $\mathrm{Im}\,B$. By dualizing the reasoning[7], since $\mathcal{Q}^\perp$ can be interpreted as a controllable subspace, the previous constructions

[7]By Proposition 2.5, given a structured pair (C,A), the orthogonal complement of the unobservable subspace is the controllable subspace of the pair $(A^\top, C^\top)$, i.e., $\mathcal{Q}^\perp = \mathcal{R}(A^\top, C^\top) =: \mathcal{O}$. Therefore there exists a fixed subspace, not necessarily $A^\top$-invariant, that we can call the *fixed observable subspace*, denoted by $\mathcal{O}^g$, s.t. $\mathcal{O}^g \subseteq \mathcal{Q}^\perp$ for almost all the realizations of the matrix $A \in \mathcal{A}$ (note that the parameter space of A and $A^\top$ coincide). As a consequence, $\exists\, \mathcal{Q}_g = (\mathcal{O}^g)^\perp$, not necessarily A-invariant, s.t. $\mathcal{Q} \subseteq \mathcal{Q}_g$ generically.

hold, thus there exists a subspace that generically contains $\mathcal{Q}$, which we can call the fixed unobservable subspace $\mathcal{Q}_g$, also spanned by canonical basis vectors, and computed as the orthogonal complement of the associated fixed subspace $\mathcal{O}^g$ of $\mathcal{O} := \mathcal{R}(A^\top, C^\top)$. It is also possible to define the maximal set of nodes contained in $\mathcal{Q}$, as the maximal subspace that is generically contained in $\mathcal{Q}$, denoted by $\mathcal{Q}^g$. This is also a fixed subspace, in the sense that it is fixed with respect to the entries of A, and it is straightforward to prove that it coincides with $\mathcal{Q}^\circ$. Moreover, it is A-invariant, while $\mathcal{Q}_g$ is not in general, since $\mathcal{Q}$ is the maximal A-invariant subspace contained in $\ker C$.

It is possible to extend the previous considerations to the subspaces $\mathcal{S}^*$ and $\mathcal{Z}^*$. In fact, given $\mathcal{S}^*$, it is possible to define the maximal set of nodes that is generically contained in $\mathcal{S}^*$, denoted by $\mathcal{S}^g$, and the minimal set of nodes that generically contains $\mathcal{S}^*$, denoted by $\mathcal{S}_g$. However, unlike for $\mathcal{R}$ (and in particular for $\mathcal{R}_g$) with respect to the A-invariance property, the subspaces $\mathcal{S}^g$ and $\mathcal{S}_g$ are not in general (C, A)-invariant. In fact, $\mathcal{S}^g$ is not in general (C, A)-invariant since $\mathcal{S}^*$ is the minimal (C, A)-invariant subspace containing $\operatorname{Im} D$. Moreover, it can be shown that the minimal (C, A)-invariant subspace that generically contains $\mathcal{S}^*$ coincides with $\mathcal{S}^\circ$, and it contains, in general, $\mathcal{S}_g$ (here in general means that the inclusion might sometimes not be strict). Similarly for $\mathcal{Z}^*$, by dualizing the reasoning, it is possible to define the minimal set of nodes that generically contains $\mathcal{Z}^*$, denoted by $\mathcal{Z}_g$, and the maximal set of nodes that is generically contained in $\mathcal{Z}^*$, denoted by $\mathcal{Z}^g$. However, unlike for $\mathcal{Q}$ (and in particular for $\mathcal{Q}^g$) with respect to the A-invariance property, the subspaces $\mathcal{Z}^g$ and $\mathcal{Z}_g$ are not in general (A, B)-invariant. In fact, $\mathcal{Z}_g$ is not in general (A, B)-invariant since $\mathcal{Z}^*$ is the maximal (A, B)-invariant subspace contained in $\ker T$. Moreover, it can be shown that the maximal (A, B)-invariant subspace that is generically contained in $\mathcal{Z}^*$ coincides with $\mathcal{Z}^\circ$, and it is contained, in general, in $\mathcal{Z}^g$. To clarify, we have the following chains of subspace inclusions, where the subspaces that always hold the property of A-invariance are colored in blue, the subspaces that always hold the property of (C, A)-invariance are colored in red, and the subspaces that always hold the property of (A, B)-invariance are colored in green:

- $\mathcal{R}^g \subseteq \mathcal{R} \subseteq \mathcal{R}_g \equiv \mathcal{R}^\circ$;
- $\mathcal{Q}^\circ \equiv \mathcal{Q}^g \subseteq \mathcal{Q} \subseteq \mathcal{Q}_g$;
- $\mathcal{S}^g \subseteq \mathcal{S}^* \subseteq \mathcal{S}_g \subseteq \mathcal{S}^\circ$;
- $\mathcal{Z}^\circ \subseteq \mathcal{Z}^g \subseteq \mathcal{Z}^* \subseteq \mathcal{Z}_g$.

To avoid misunderstandings, we refer to the sets of nodes $\mathcal{R}^g$, $\mathcal{Q}_g$, $\mathcal{S}^g$, $\mathcal{S}_g$, $\mathcal{Z}^g$, $\mathcal{Z}_g$ as the fixed subspaces (not necessarily invariant) contained or containing generically the subspaces $\mathcal{R}$, $\mathcal{Q}$, $\mathcal{S}^*$, $\mathcal{Z}^*$, as pointed above. On the other hand, the sets of nodes $\mathcal{R}^\circ$, $\mathcal{Q}^\circ$, $\mathcal{S}^\circ$, $\mathcal{Z}^\circ$ are denoted as the minimal/maximal invariant node sets containing or contained generically in the subspaces $\mathcal{R}$, $\mathcal{Q}$, $\mathcal{S}^*$, $\mathcal{Z}^*$, respectively.

Remark 2.28 (Clarifying the confusion arising by the term "fixed" subspace). As for $\mathcal{R}$, $\mathcal{R}^\circ$ and $\mathcal{Q}$, $\mathcal{Q}^\circ$, the subsets $\mathcal{S}^\circ$ and $\mathcal{Z}^\circ$ are, respectively, the minimal (C, A)-

invariant node set containing $\mathcal{S}^*$ and the the maximal (A, B)-invariant node set contained in $\mathcal{Z}^*$. Those are sets of nodes, meaning that the generators of their corresponding subspaces are independent of (or "*fixed*" w.r.t.) the entries of A. It is important to highlight that the term *fixed* has been introduced in [1] to define the subspaces $\mathcal{S}_g$ and $\mathcal{Z}^g$ (which in general are not conditioned nor controlled A-invariant) that are essential for the structural analysis of the DDP, and not for the node sets $\mathcal{S}^\circ$ and $\mathcal{Z}^\circ$. In fact, in [1] it was proven that if the DDP solvability conditions (in particular, DDPSF and DDPDF) are solved for the subspaces $\mathcal{S}_g$ and $\mathcal{Z}^g$ (which are the minimal (resp. maximal) subspaces generated by constant vectors that generically contain (resp. are contained in) $\mathcal{S}^*$ (resp. $\mathcal{Z}^*$)), then the DDP is generically solvable. Those results do not require the definition of the node sets $\mathcal{S}^\circ$ and $\mathcal{Z}^\circ$, since the considerations are about generic set inclusions, and do not explicitly require the fixed subspaces $\mathcal{S}_g$ and $\mathcal{Z}^g$ to be conditioned (resp. controlled) A-invariant. In fact, the terms *fixed* for $\mathcal{S}_g$ and $\mathcal{Z}^g$ denote that: 1) Those subsets are generated by canonical basis vectors (thus are in one-to-one correspondence with sets of nodes, as for $\mathcal{S}^\circ$ and $\mathcal{Z}^\circ$); 2) Given a structured matrix A, the DDP conditions on $\mathcal{S}_g$ and $\mathcal{Z}^g$ imply those on $\mathcal{S}^*$ and $\mathcal{Z}^*$ (as for $\mathcal{S}^\circ$ and $\mathcal{Z}^\circ$), while the failure of the DDP conditions on $\mathcal{S}_g$ and $\mathcal{Z}^g$ allows solvability of the DDP conditions on $\mathcal{S}^*$ and $\mathcal{Z}^*$ only on a proper variety of the parameter space of A. Note that this last property does not hold in general for $\mathcal{S}^\circ$ and $\mathcal{Z}^\circ$. The term "fixed" has been introduced, with similar intentions, to define the fixed controllable subspace $\mathcal{R}^g$ in [13], which is the subset of $\mathcal{R}$ that is controllable for almost all realizations $A \in \mathcal{A}$ (although, in general not A-invariant).

In this thesis, the fixed controllable subspace $\mathcal{R}^g$ has been used to prove controllability of a temporal network in Paper A, while the minimal/maximal conditioned/controlled A-invariant node sets $\mathcal{S}^\circ$ and $\mathcal{Z}^\circ$ are the subspaces used in the disturbance decoupling formulation over networks addressed in Paper B (and in Paper C).

It is possible to construct the sets $\mathcal{Z}^g$ and $\mathcal{S}_g$ by means of graph linkings arising from a structured transfer matrix approach to the DDP, where conditions are written in terms of the structure at infinity of specif transfer functions involving the control inputs, the disturbances and the targets (for details, we refer to Sections 7.3 and 7.4 of [36] and to [37–40]).

To clarify further the reason why the structural DDP analysis in [1] is carried out on $\mathcal{Z}^g$ we give the following fundamental result for $\mathcal{Z}^*$. A similar result applies for $\mathcal{S}_g$ as well.

Proposition 2.17 (Proposition 7 of [1]). *Consider A, B, T structured matrices, and let $\mathcal{Z}^*$ be the maximal structured (A, B)-invariant subspace contained in* $\ker T$[8]. *Let x be a linear constant combination of canonical vectors, i.e., $x = \sum_{i=1}^{n} x_i e_i$, $x_i \in \mathbb{R}$. $x \in \mathcal{Z}^*$ generically (relatively to the parameter set of A) if and only if $e_i \in \mathcal{Z}^*$ generically for all $i \in \{1, \ldots, n\}$ s.t. $x_i \neq 0$.*

[8]The maximal structured (A, B)-invariant subspace contained in $\ker T$, $\mathcal{Z}^*$, depends on the parameter values and contains a fixed part which is $\mathcal{Z}^g$.

Remark 2.29. The consequence is obvious: if we consider a structured system, the set where $x \in \mathcal{Z}^*$ but $x \notin \mathcal{Z}^g$ has measure zero, since those vectors x do not belong to $\mathcal{Z}^*$ generically. Therefore, the situations where the DDP fails with $\mathcal{Z}^g$ (and similarly with $\mathcal{S}_g$) but has success with $\mathcal{Z}^*$ (resp. $\mathcal{S}^*$), has zero measure in the space of the free entries of $A \in \mathcal{A}$.

It is important to highlight the key difference between the structural approach to the DDP and the node set formulation given in this thesis (in particular in Paper B): since the node set $\mathcal{Z}^\circ$ (resp. $\mathcal{S}^\circ$) is controlled (resp. conditioned) A-invariant it is possible to design a state feedback (resp. an output injection) friend of $\mathcal{Z}^\circ$ (resp. of $\mathcal{S}^\circ$). In fact, the synthesis of the state, output and dynamical feedback relies on the sets $\mathcal{Z}^\circ$, $\mathcal{S}^\circ$ being controlled and conditioned A-invariant. We conclude this subsection with a clarifying example.

Example 2.9

Consider the networks depicted in Figure 2.13 and their edge-induced adjacency matrices $A = (a_{ij})$.

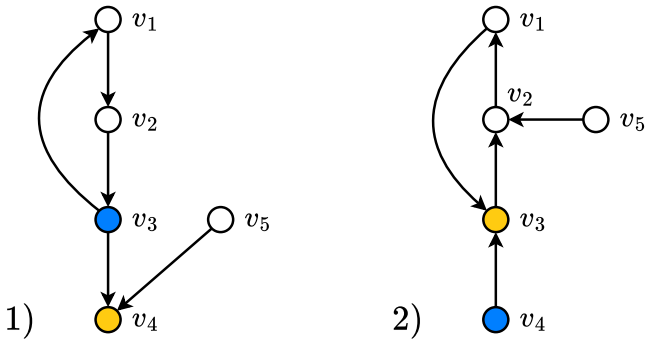


Figure 2.13: *Targets are in yellow, control inputs in blue (Case* 1*)* : $\mathcal{B} = \{v_3\}$, $\mathcal{T} = \{v_4\}$*; Case* 2*)* : $\mathcal{B} = \{v_4\}$, $\mathcal{T} = \{v_3\}$*).*

Case 1): Using the recursions of Propositions 2.12 and 2.14, one gets $\mathcal{Z}^* = \text{span}\{e_1, e_2, a_{45}e_3 - a_{43}e_5\}$, and $\mathcal{Z}^\circ = \{v_1, v_2\}$. For this example $\mathcal{Z}^\circ = \mathcal{Z}^g$, since it is the maximal subspace that is generated by constant vectors that is generically contained in $\mathcal{Z}^*$. The subspace $\mathcal{Z}_g = \{v_1, v_2, v_3, v_5\}$ is the minimal subspace generated by constant vectors that generically contains $\mathcal{Z}^*$.
Case 2): We note that this network corresponds to the transposed of the network depicted in Figure 2.10 of Example 2.7. In Example 2.7, from the graph of Figure 2.10, with $\mathcal{D} = \{v_3\}$ and $\mathcal{C} = \{v_4\}$, we have derived $\mathcal{S}^* = \text{span}\{e_3, a_{13}e_1 + a_{43}e_4\}$ and $\mathcal{S}^\circ = \mathcal{V}$. We note that $\mathcal{S}^g = \{v_3\}$ and $\mathcal{S}_g = \{v_1, v_3, v_4\} \subset \mathcal{S}^\circ$ are, respectively, the maximal and the minimal subspaces generated by constant vectors that generically belong and contain the subspace $\mathcal{S}^*$. It is easy to see that they are not conditional A-invariant. Using Proposition 2.16, the subsets $\mathcal{Z}^*$ and $\mathcal{Z}^\circ$ on the transposed graph $\mathcal{G}(A^\top)$ of Figure 2.13 can be computed by taking the orthogonal complements of $\mathcal{S}^*$ and $\mathcal{S}^\circ$. We therefore obtain $\mathcal{Z}^* = \text{span}\{e_2, e_5, a_{13}e_4 - a_{43}e_1\}$

and $\mathcal{Z}^\circ = \emptyset$. We note that $\mathcal{Z}_g = \{v_1, v_2, v_4, v_5\}$ and $\mathcal{Z}^g = \{v_2, v_5\} \supset \mathcal{Z}^\circ$, and that they are not controlled A-invariant.

Example 2.9 shows that given the sets $\mathcal{T}$, $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$, the quantities $\dim(\mathcal{Z}^*) - |\mathcal{Z}^\circ|$ and $|\mathcal{S}^\circ| - \dim(\mathcal{S}^*)$ depend on the underlying network topology. This is a subject of future investigation.

3

Compendium on the disturbance decoupling problem

This chapter contains an extension of the contents of Paper B, namely, the disturbance decoupling problem over networks with pole placement. The contents of this section are not a background material for Papers A, B and C, but rather a starting point of future research directions still to be completed.

3.1 Disturbance decoupling problem with stability over networks

This section covers the disturbance decoupling problem with stability (or, more in general, pole placement) over networks, via state feedback. Similar considerations can be made for the DDPOF and DDPDF with stability. We recommend to read this section after having read Paper B. The contents of this section are based on the results of Paper B and on geometric control results that can be found in [8–10, 29–31, 41–45].

The DDPSF over networks with stability (**DDPSF-S**) is solved by selecting $\mathcal{B}$ s.t.

1. $\mathcal{D} \subseteq \mathcal{Z}^\circ(\mathcal{B})$ and $(A - BF)\mathcal{Z}^\circ \subseteq \mathcal{Z}^\circ$ (this problem is solved by minimizing $|\mathcal{B}|$ in Problem 2A of Paper B);
2. $A - BF$ is Hurwitz.

Remark 3.1 (External and internal stability of $\mathcal{Z}^\circ$). The requirement of $A - BF$ being Hurwitz corresponds of asking that the friend F of $\mathcal{Z}^\circ(\mathcal{B})$ both internally and externally stabilizes $\mathcal{Z}^\circ$. As in Definition 2.16, internal stability of an $(A - BF)$-invariant corresponds of asking that $\forall\, x_0 \in \mathcal{Z}^\circ$ the corresponding trajectories asymptotically converge to the origin. Note that, independent of the internal

stability, the condition $\mathcal{D} \subseteq \mathcal{Z}^\circ$ guarantees that $\forall\, x_0 \in \mathcal{Z}^\circ$ the corresponding trajectories (may they be stable or unstable) are invisible to the targets. External stability of an $(A - BF)$-invariant corresponds of asking that $\forall\, x_0 \in \mathbb{R}^n$ the corresponding trajectories asymptotically converge to $\mathcal{Z}^\circ$, thus, independent of the internal stability, all trajectories are asymptotically invisible to the targets. The solvability of the DDPSF-S translates in asking that $\forall\, x_0 \in \mathbb{R}^n$ the corresponding trajectories converge asymptotically to the origin, while ensuring that the dynamics excited by the disturbance remains invisible to the targets.

One of the main assumptions that are made in Paper B is that it is always possible to add self-loops with sufficiently negative weights to the network nodes to ensure diagonal dominance and therefore asymptotic stability. This requirement, under the assumption that the disturbances do not destabilize the zero-dynamics of the system (see Theorem 3.4), is sufficient for solving the DDPSF-S, but can be relaxed and narrowed down if one allows a different choice of the input nodes set $\mathcal{B}$. In fact, the optimal solution $\mathcal{B}_{\text{opt}}$ of Problem 2A is s.t. $\mathcal{B}_{\text{opt}} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}_{\text{opt}}), A)$, thus from Theorem 8 of Paper B the unique friend of $\mathcal{Z}^\circ$ is $F = B^\top A + F_p$ up to $F_p = M(Z^\perp)^\top$, with M being a $m \times (n - |\mathcal{Z}|)$ matrix of coefficients, $Z^\perp$ a basis matrix of $\mathcal{Z}^{\circ\perp}$ and B the input matrix of elementary columns associated to $\mathcal{B}_{\text{opt}}$. As it will soon appear clear, the degree of freedom in the choice of F_p can be used to address the external stability of $\mathcal{Z}^\circ$ but cannot influence its internal stability. If we denote by Z a basis matrix for $\mathcal{Z}^\circ$, from Proposition 2.10 one notes that since F is a friend of $\mathcal{Z}^\circ$ there exists X, U s.t. $AZ = ZX + BU = [Z \quad B][X^\top \quad U^\top]^\top$. Therefore,

$$[X^\top \quad U^\top]^\top = [Z \quad B]^\dagger AZ + \Gamma\alpha, \tag{3.1}$$

where $[Z \quad B]^\dagger$ is the pseudo-inverse of $[Z \quad B]$, Γ is a matrix whose columns form a basis for $\ker[Z \quad B]$ and α is a column vector of linear combinators of a generic element of the null space of $[Z \quad B]$. Now note that $\mathcal{B}_{\text{opt}} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}_{\text{opt}}), A) \implies \mathcal{B}_{\text{opt}} \cap \mathcal{Z}^\circ(\mathcal{B}_{\text{opt}}) = \emptyset$, thus $\ker[Z \quad B] = \{0\}$. This means that the degree of freedom $\Gamma\alpha$ cannot be exploited for the internal stability of $\mathcal{Z}^\circ$ if $\mathcal{B} = \mathcal{B}_{\text{opt}}$, or more generally, when $\mathcal{B}$ corresponds to the out-border of $\mathcal{Z}^\circ(\mathcal{B})$. To clarify further, consider $u = -B^\top Ax + u'$, with $u' = -F_p x$, and $\mathcal{B} = \mathcal{B}_{\text{opt}}$ s.t. $F = B^\top A + F_p$ and note that since $\mathcal{Z}^\circ(\mathcal{B}_{\text{opt}})$ is $(A - BF)$-invariant and, in particular $(A - BB^\top A)$-invariant, we can perform a change of basis as in Remark 2.7.

Taking $T = [Z\ T_2]$, where T_2 completes the basis in $\mathbb{R}^n$, gives

$$T^{-1}(A - BB^\top A)T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}. \tag{3.2}$$

The block triangular structure of $T^{-1}(A - BB^\top A)T$ is a consequence of $\mathcal{Z}^\circ$ being $(A - BB^\top A)$-invariant, that is achieved by cutting the edges entering the out-border of $\mathcal{Z}^\circ(\mathcal{B})$. Since $\mathcal{Z}^\circ$ is also $(A - BF)$-invariant, then F_p is s.t. $T^{-1}(A - BB^\top A - BF_p)T$ is also a block upper triangular matrix. We conclude that $u' = -F_p x$ can externally stabilize $\mathcal{Z}^\circ$ if (A_{22}, B_2) is controllable, or if the pair (A_{22}, B_2) is stabilizable (see Remark 2.8). We can summarize those considerations in the following Theorem.

Theorem 3.1. *Assume $\mathcal{B}$ is a set of input nodes s.t. $\mathcal{B} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$ so that $F = B^\top A + F_p$, with $F_p = M(Z^\perp)^\top$, is the family of friends of $\mathcal{Z}^\circ(\mathcal{B})$. W.r.t. the change of basis in equation* (3.2) *we have the following:*

- *$\exists\, F_p$ s.t. $\mathcal{Z}^\circ(\mathcal{B})$ is externally stable $\iff$ (A_{22}, B_2) is stabilizable;*
- *$\mathcal{Z}^\circ(\mathcal{B})$ is internally stable $\iff$ A_{11} is Hurwitz.*

Remark 3.2 (Fixed external eigenstructure of $\mathcal{Z}^\circ$). An optimal solution to Problem 2A does not internally stabilize $\mathcal{Z}^\circ$; therefore, additional control inputs acting within $\mathcal{Z}^\circ$ are required to ensure internal stability. The degree of freedom in F_p can be exploited for the external stabilization of $\mathcal{Z}^\circ$. Interpreting $\mathcal{Z}^\circ$ as a vector subspace, if the pair (A_{22}, B_2) is not controllable it means that F_p can be introduced to assign in the quotient space $\mathbb{R}^n/\mathcal{Z}^\circ$ only the eigenvalues of the linear map $A - BB^\top A$ restricted to $(\mathcal{R} + \mathcal{Z}^\circ)/\mathcal{R}$ but not those in the restriction $\mathbb{R}^n/(\mathcal{R} + \mathcal{Z}^\circ)$, where $\mathcal{R}$ denotes the controllable subspace of the pair $(A - BB^\top A, B)$ defined by the recursion in Theorem 2.3. The eigenvalues of $A - BB^\top A$ restricted to $\mathbb{R}^n/(\mathcal{R} + \mathcal{Z}^\circ)$ are not accessible by F_p, and are thus called the fixed external eigenstructure of $\mathcal{Z}^\circ$, denoted by $\sigma^{\text{ext}}_{\text{fixed}}(\mathcal{Z}^\circ)$. The pair (A_{22}, B_2) is stabilizable iff the eigenvalues in $\sigma^{\text{ext}}_{\text{fixed}}(\mathcal{Z}^\circ)$ are stable. For more details on the external stabilization of an invariant subspace, we refer to [10, 46, 47].

In the hypothesis of Theorem 3.1, we propose a way to structurally guarantee that the pair (A_{22}, B_2) is stabilizable or, if possible, controllable, for almost all $A \in \mathcal{A}$, by introducing a minimal set of self-loops with sufficiently negative weights in some specific nodes in $\mathcal{Z}^{\circ\perp}$.

To this end, denote by $\mathcal{R}^\circ$ the minimal input-containing $(A - BB^\top A)$-invariant node set, defined in Proposition 2.3. Moreover, consider the structured pair $(A - BB^\top A, B)$, $A \in \mathcal{A}$, and referring to the change of basis in equation (3.2) define the matrix $A_{\text{dec}} = T\,\text{blkdiag}(A_{11}, A_{22})\,T^{-1}$ so that we can construct and auxiliary graph $\mathcal{G}(A_{\text{dec}})$ consisting of two connected components identified by the node sets $\mathcal{Z}^\circ$ and $\mathcal{Z}^{\circ\perp}$. We can write $\mathcal{G}(A_{\text{dec}}) = \mathcal{G}(\mathcal{Z}^\circ, \mathcal{E}_{\mathcal{Z}^\circ}) \cup \mathcal{G}(\mathcal{Z}^{\circ\perp}, \mathcal{E}_{\mathcal{Z}^{\circ\perp}})$, where $\mathcal{E}_{\mathcal{Z}^\circ}$ and $\mathcal{E}_{\mathcal{Z}^{\circ\perp}}$ denote the original edge sets between the nodes in $\mathcal{Z}^\circ$ and $\mathcal{Z}^{\circ\perp}$, respectively. Note that the edges from $\mathcal{Z}^\circ$ to $\mathcal{Z}^{\circ\perp}$ have been removed by the friend $B^\top A$, while having set $A_{12} = 0$ in equation (3.2), i.e., having imposed the block-diagonal structure $\text{blkdiag}(A_{11}, A_{22})$, corresponds to the removal of the edges from $\mathcal{Z}^{\circ\perp}$ to $\mathcal{Z}^\circ$. Note that those edges are not removed in the original graph of $\mathcal{G}(A - BB^\top A)$ and so $\mathcal{B} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}_{\text{opt}}), A)$ can reach nodes in $\mathcal{Z}^\circ$. However, the constraint that F_p must be of the form $M(Z^\perp)^\top$, which is necessary to guarantee the $(A - BB^\top A - BF_p)$-invariance of $\mathcal{Z}^\circ$, reflects the fact that controllability from $\mathcal{B}$ can only affect nodes that are in $\mathcal{Z}^{\circ\perp}$. In agreement with Theorem 2.5, denote by $\mathcal{L}$ the maximal disjoint union of cycles and stems rooted in the input nodes $\mathcal{B} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$ in $\mathcal{G}(\mathcal{Z}^{\circ\perp})$. Note that $|\mathcal{L}|$ corresponds to the generic dimension of the controllable subspace of the structural pair (A_{22}, B_2).

A way to structurally guarantee that the pair (A_{22}, B_2) is stabilizable for almost all $A \in \mathcal{A}$ is by introducing self-loops with sufficiently negative weights in the

nodes belonging to the sets:

1. $\mathcal{Z}^{\circ\perp} \setminus (\mathcal{R}^\circ \cap \mathcal{Z}^{\circ\perp})$ (set of nodes in $\mathcal{Z}^{\circ\perp}$ not reached by $\mathcal{B} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$);
2. $(\mathcal{R}^\circ \cap \mathcal{Z}^{\circ\perp}) \setminus \mathcal{L}$ (minimal set of nodes in $\mathcal{Z}^{\circ\perp}$ that are not structurally controllable).

Note that if $\mathcal{Z}^{\circ\perp} \setminus (\mathcal{R}^\circ \cap \mathcal{Z}^{\circ\perp}) = \emptyset$, placing the self-loops in $(\mathcal{R}^\circ \cap \mathcal{Z}^{\circ\perp}) \setminus \mathcal{L}$ renders the pair (A_{22}, B_2) structurally controllable and the poles arbitrarily assignable via F_p. It follows that the self-loops do not need to be of negative weight in this case.

The following example provides a graphical illustration of the underlying intuition.

Example 3.1

Consider the network depicted in Figure 3.1, whose computation of $\mathcal{Z}^\circ$ has been shown in Figure 2.6.

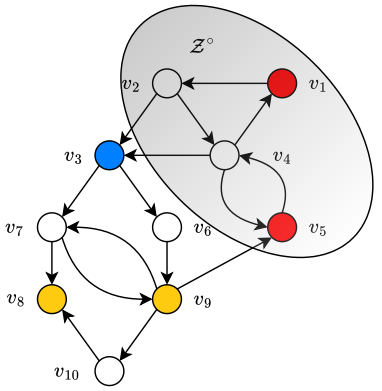


Figure 3.1: *Target nodes are in yellow, disturbances are in red, control inputs in blue.*

$\mathcal{B} = \mathcal{B}_{\text{opt}} = \{v_3\}$ and the friend $B^\top A$ of $\mathcal{Z}^\circ$ has the effect of removing in closed loop the edges (v_2, v_3) and (v_4, v_3). Note that in the computation of $\mathcal{L}$ the edge (v_9, v_5) must be removed. The auxiliary graph $\mathcal{G}(A_{\text{dec}})$ is depicted in Figure 3.2. We note that $\mathcal{Z}^{\circ\perp} \setminus (\mathcal{R}^\circ \cap \mathcal{Z}^{\circ\perp}) = \emptyset$ and that $\mathcal{L} = \{v_3, v_6, v_9, v_{10}, v_8\}$. Adding a self-loop to node v_7 is sufficient to guarantee the external stabilization of $\mathcal{Z}^\circ$, in agreement with Theorem 3.1. The numerical entries of F_p depend on the entries of the pair (A_{22}, B_2) and on the poles' location. The action of the friend $F = B^\top A + F_p$ in closed loop is depicted in Figure 3.3.

As already mentioned, it is not possible to act on the design of F to internally stabilize $\mathcal{Z}^\circ(\mathcal{B})$ if $\mathcal{B} = \partial_+\big(\mathcal{Z}^\circ(\mathcal{B}), A\big)$. In the following, we augment $\mathcal{B}_{\text{opt}} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}_{\text{opt}}), A)$ with additional control inputs acting within $\mathcal{Z}^\circ$.

Assume that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, with $\mathcal{B}_2 = \mathcal{B}_{\text{opt}}$ s.t. $\mathcal{D} \subseteq \mathcal{Z}^\circ(\mathcal{B}_2)$ and where $\mathcal{B}_1 \cap \mathcal{B}_{\text{opt}} = \emptyset$. From the recursion in Proposition 2.14 it is follows that $\mathcal{B}_1 \subseteq \mathcal{Z}^\circ$, since

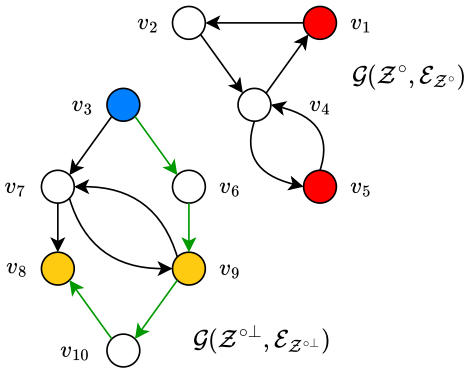


Figure 3.2: *The auxiliary graph $\mathcal{G}(A_{dec}) = \mathcal{G}(\mathcal{Z}^\circ, \mathcal{E}_{\mathcal{Z}^\circ}) \cup \mathcal{G}(\mathcal{Z}^{\circ\perp}, \mathcal{E}_{\mathcal{Z}^{\circ\perp}})$. The path that identifies the set $\mathcal{L}$ is colored in green. Target nodes are in yellow, disturbances are in red, control inputs in blue.*

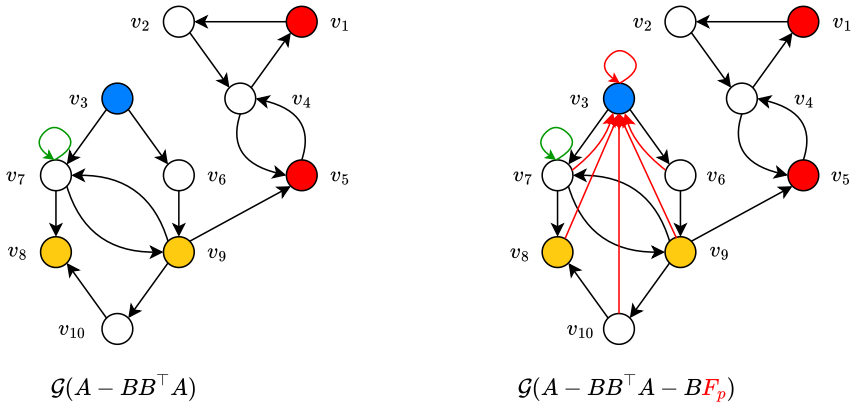


Figure 3.3: *The action of the externally stabilizing friend $B^\top A + F_p$ of $\mathcal{Z}^\circ$ in closed loop, with the additional self-loop in v_7, colored in green. The friend $B^\top A$ removes the edges (v_2, v_3) and (v_4, v_3), while F_p adds edges from $\mathcal{Z}^{\circ\perp}$ onto $\mathcal{B}$ of arbitrary weight, colored in red.*

$\mathcal{B}_{\text{opt}} = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$ implies $\mathcal{Z}^\circ(\mathcal{B}) \equiv \mathcal{Z}^\circ(\mathcal{B}_{\text{opt}})$. The input matrix of elementary columns associated with $\mathcal{B}$ is $B = [B_1\ B_2]$ s.t. $\operatorname{Im} B_1 = \mathcal{B}_1$ and $\operatorname{Im} B_2 = \mathcal{B}_2$, thus $A - BF = A - B_1F_1 - B_2F_2$, where $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ is the friend of $\mathcal{Z}^\circ$ to be designed. We exploit Theorem 8 of Paper B to write $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ B_2^\top A + F_p \end{bmatrix} = F_a + F_b$, where $F_a = \begin{bmatrix} 0 \\ B_2^\top A \end{bmatrix}$ and $F_b = \begin{bmatrix} F_1 \\ F_p \end{bmatrix}$, with $F_p = M(Z^\perp)^\top$. We know that $B_2^\top A$ is a friend of $\mathcal{Z}^\circ(\mathcal{B}_2) \equiv \mathcal{Z}^\circ(\mathcal{B})$, thus F_a is. Now, note that $\mathcal{Z}^\circ$ is $(A - BF_a)$-invariant, therefore using the change of basis as in equation (3.2) we obtain the same block

upper triangular form. We already proved that F_p can be exploited for the external stabilization of $\mathcal{Z}^\circ$. Now note that since $\mathcal{B}_1 \cap \mathcal{Z}^\circ \neq \emptyset$, we can exploit the degree of freedom $\Gamma\alpha$ in equation (3.1) in the design of F_1 for the internal stabilization of $\mathcal{Z}^\circ$. By using the change of basis as in equation (3.2) we get

$$T^{-1}(A - BF_a)T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad T^{-1}B = T^{-1}[B_1\ B_2] = \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & \hat{B}_2 \end{bmatrix}. \tag{3.3}$$

We already noted in Theorem 3.1 that there exists an externally stabilizing F_p for $\mathcal{Z}^\circ$ iff $(A_{22}, \hat{B}_2)$ is stabilizable and that we need A_{11} to be Hurwitz in order to internally stabilize $\mathcal{Z}^\circ$. However now we have an additional degree of freedom, due to the control inputs $\mathcal{B}_1$ within $\mathcal{Z}^\circ$. In fact, we conclude that, w.r.t. equations (3.3), if the pair $(A_{11}, \hat{B}_1)$ is controllable then $\mathcal{Z}^\circ$ is internally stabilizable. If the pair $(A_{11}, \hat{B}_1)$ is stabilizable we have that $\mathcal{Z}^\circ$ is internally stabilizable iff the disturbance vector Dw entering the state equation in the system (2.1), which satisfies $\mathcal{D} \subseteq \mathcal{Z}^\circ$, does not excite the zero-dynamics with unstable modes. In order to explain this concept, we need to define the maximal controllable subspace within $\mathcal{Z}^\circ$.

The following definition is attributed to Wonham and Morse (see Theorem 4.3 of [30]).

Definition 3.1 (Controllable subspace within a subspace). Given an $(A - BF)$-invariant subspace $\mathcal{Z}$, the controllable subspace within $\mathcal{Z}$, denoted by $\mathcal{R}_\mathcal{Z}$, is defined as the controllable subspace of the pair $(A - BF, B_1)$, where B_1 is s.t. $\operatorname{Im} B_1 = \operatorname{Im} B \cap \mathcal{Z}$. If $\mathcal{Z} = \mathcal{Z}^*(A, B; \ker T)$, then $\mathcal{R}_{\mathcal{Z}^*}$ is called the maximal target-nulling controllable subspace.

The following definition corresponds to Definition 4.1.3 of [10].

Definition 3.2 (Self-bounded subspace). Let $\mathcal{X} \subseteq \mathbb{R}^n$ be an (A, B)-controlled invariant subspace contained in $\ker T$. The subspace $\mathcal{X}$ is said to be self-bounded with respect to $\ker T$ if

$$\mathcal{Z}^*(A, B; \ker T) \cap \operatorname{Im} B \subseteq \mathcal{X},$$

where $\mathcal{Z}^*(A, B; \ker T)$ is the maximal target-nulling (A, B)-invariant subspace defined by the recursion of Proposition 2.12.

Remark 3.3. It has been proven in Lemma 1.1 of [48] that $\mathcal{R}_{\mathcal{Z}^*} = \mathcal{Z}^*(A, B; \ker T) \cap \mathcal{S}^*(T, A; \operatorname{Im} B)$. As extensively described in Sections 4.1.2 and 4.1.3 of [10], the subspace $\mathcal{R}_{\mathcal{Z}^*}$ is the minimal (A, B)-invariant subspace self-bounded w.r.t. $\ker T$. In the set of all possible (A, B)-invariant subspaces, self-bounded w.r.t. $\ker T$, the subspaces $\mathcal{R}_{\mathcal{Z}^*}$ and $\mathcal{Z}^*(A, B; \ker T)$ are, respectively, the infimum and the supremum, and the former is the subspace that allows the maximal degree of freedom in the design of the friend F (see, e.g., Lemma 2.5 of [42] and [49]).

In the following, Definition 3.1 is adapted and reinterpreted in the context of networks.

Definition 3.3 (Maximal controllable subspace in $\mathcal{Z}^\circ$). Given the $(A - BF_a)$-invariant subspace $\mathcal{Z}^\circ(\mathcal{B})$, the maximal controllable subspace within $\mathcal{Z}^\circ(\mathcal{B})$, denoted by $\mathcal{R}_{\mathcal{Z}^\circ}$, is defined as the controllable subspace of the pair $(A - BF_a, B_1)$, where B_1 is an input matrix with elementary columns s.t. $\operatorname{Im} B_1 = \mathcal{B}_1 = \mathcal{B} \cap \mathcal{Z}^\circ$.

Remark 3.4. The subspace $\mathcal{R}_{\mathcal{Z}^\circ}$ is (A, B)-invariant, since it is $(A - BF_a)$-invariant.

Similarly, Definition 3.2 is adapted to our set of nodes of interest, $\mathcal{Z}^\circ$. We first need the following Lemma.

Lemma 3.1. *If $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, $\mathcal{B}_2 = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$ and $\mathcal{B}_1 \subseteq \mathcal{Z}^\circ$, then $\mathcal{B}_1 = \mathcal{B} \cap \mathcal{Z}^\circ = \mathcal{B} \cap \mathcal{Z}^*$.*

Proof: By Theorem 2.9, $\mathcal{Z}^\circ \subseteq \mathcal{Z}^*$, and since $\mathcal{B}$ is a set of nodes, $\mathcal{B} \cap \mathcal{Z}^* = \mathcal{B}_1 \cup \hat{\mathcal{B}}$, with $\hat{\mathcal{B}} \subseteq \mathcal{B}_2$. However, for each $v_i \in \mathcal{B}_2 = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$ there $\exists$ a path $\mathcal{P}^{v_i, v_t} = \{v_i, v_{i+1}, \dots, v_t\}$ from $v_i \in \mathcal{V}$ to $v_t \in \mathcal{T}$ of length $\leq \ell_i$, and s.t. $\mathcal{P}^{v_{i+1}, v_t} \cap \mathcal{B} = \emptyset$. By Lemma 2 of Paper B this means that $\mathcal{B}_2 \subseteq \mathcal{Z}^{\circ\perp}$. Similarly, the existence of the path $\mathcal{P}^{v_i, v_t}$ of length ℓ_i from $v_i \in \mathcal{B}_2 = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$ to $\mathcal{T}$ can be used to prove that $e_i \notin \mathcal{Z}^*$. First, note that for every node $v_i \in \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$ there must exist a path $\mathcal{P}^{v_i, v_t}$ such that $\mathcal{P}^{v_{i+1}, v_t} \cap \mathcal{B} = \emptyset$, meaning that the vector Ae_i has at least one nonzero component not in $\operatorname{Im} B$. Now consider all the nodes in $\mathcal{B}_2$ and sort them according to the lengths of their shortest path to $\mathcal{T}$ of the kind $\mathcal{P}^{v_i, v_t}$ s.t. $\mathcal{P}^{v_{i+1}, v_t} \cap \mathcal{B} = \emptyset$, and call those lengths ℓ_i, $i \in \{1, \dots, |\mathcal{B}_2|\}$, $\ell_i \leq n \ \forall i$. Consider the nodes in $\mathcal{B}_2$ s.t. $\ell_i = 0$. By the existence of the path it follows that $e_i \in \operatorname{Im} T$. Take such e_i in $\operatorname{Im} T$ and note that, from the recursion of Proposition 2.12, $e_i \notin \ker T = \mathcal{Z}_0$. Consider now the nodes in $\mathcal{B}_2$ s.t. $\ell_i = 1$. Take such e_i in $\mathcal{Z}_0$ (the nodes not in $\mathcal{Z}_0$ have a nonzero component in $\operatorname{Im} T$, thus are s.t. $\ell_i = 0$) and since $\ell_i = 1$ it follows that $Ae_i \notin \mathcal{Z}_0$ since Ae_i has a nonzero component in $\operatorname{Im} T$. Now, note that since $\mathcal{P}^{v_{i+1}, v_t} \cap \mathcal{B} = \emptyset$, $Ae_i \notin \operatorname{Im} B$ (i.e., $\exists$ at least a component in Ae_i not in $\operatorname{Im} B$). From the recursion of Proposition 2.12 it follows that $e_i \notin \mathcal{Z}_1$. Consider now the nodes in $\mathcal{B}_2$ s.t. $\ell_i = 2$, that is, take $e_i \in \mathcal{Z}_1$ (i.e., s.t. $Ae_i \in \mathcal{Z}_0 + \operatorname{Im} B$), and s.t. $Ae_i \notin \mathcal{Z}_1$. $Ae_i \notin \mathcal{Z}_1$ means that $A(Ae_i) = A^2 e_i \notin \mathcal{Z}_0 + \operatorname{Im} B$, which holds since $\ell_i = 2$ and by the path assumption. In fact $A^2 e_i$ has a nonzero component in $\operatorname{Im} T$ and at least a component not in $\operatorname{Im} B$. Moreover, by the path assumption, $Ae_i \notin \operatorname{Im} B$, thus $Ae_i \notin \mathcal{Z}_1 + \operatorname{Im} B \implies e_i \notin \mathcal{Z}_2$. By iterating the reasoning, consider the nodes in $\mathcal{B}_2$ s.t. $\ell_i = \ell$. Take $e_i \in \mathcal{Z}_{\ell-1}$ s.t. $Ae_i \notin \mathcal{Z}_\ell$, i.e., $A^2 e_i \notin \mathcal{Z}_{\ell-1} + \operatorname{Im} B$ since $A^\ell e_i \notin \mathcal{Z}_0 + \operatorname{Im} B$. By the path assumption $Ae_i \notin \operatorname{Im} B$, thus $e_i \notin \mathcal{Z}_\ell$. When $\ell = \max_i\{\ell_i\}$, we conclude that $e_i \notin \mathcal{Z}_m$, $m \geq \ell$, $\forall\, e_i \in \mathcal{B}_2$, which implies that $e_i \notin \mathcal{Z}^*$, since the sequence in Proposition 2.12 is strictly decreasing (see, e.g., Theorem 4.10 of [31]). ∎

Theorem 3.2. *The subspace $\mathcal{R}_{\mathcal{Z}^\circ}$ of Definition 3.3 is self-bounded w.r.t.* $\ker T$.

Proof: By definition, $\mathcal{R}_{\mathcal{Z}^\circ}$ is the controllable subspace of the pair $(A - BF_a, B_1)$, therefore it is the minimal $(A - BF_a)$-invariant subspace containing $\operatorname{Im} B_1$, i.e., $\operatorname{Im} B_1 = \mathcal{B}_1 = \mathcal{Z}^\circ \cap \mathcal{B} \subseteq \mathcal{R}_{\mathcal{Z}^\circ}$. By Lemma 3.1, the thesis follows. ∎

In the following Proposition, an alternative formula to compute $\mathcal{R}_{\mathcal{Z}^\circ}$ is provided.

Proposition 3.1. *The controllable subspace $\mathcal{R}_{\mathcal{Z}^\circ}$ of the pair $(A - BF_a, B_1)$, as in Definition 3.3, can be computed by $\mathcal{Z}^\circ(A, \mathcal{B}; \mathcal{T}^\perp) \cap \mathcal{S}^*(T, A; \operatorname{Im} B)$, where $\mathcal{S}^*(T, A; \operatorname{Im} B)$ is the minimal input-containing (T, A)-invariant subspace, defined by the recursion of Proposition 2.15.*

Proof: W.r.t. equation (3.3) we note that in closed loop $\mathcal{B}_1$ can only influence $\mathcal{Z}^\circ$ while $\mathcal{B}_2$ can influence $\mathcal{Z}^\circ$ and $\mathcal{Z}^{\circ\perp}$. In fact, by the construction of $\mathcal{Z}^\circ \subseteq \ker T$ and by the fact that $\mathcal{B}_2 = \partial_+\big(\mathcal{Z}^\circ(\mathcal{B}), A\big)$ it follows that $\mathcal{B}_2$ is the only control input set that reaches $\mathcal{T} \subseteq \mathcal{Z}^{\circ\perp}$. This means that $\forall\, v_i \in \mathcal{B}_2$ there must exist a path $\mathcal{P}^{v_i, v_j}$ for some $v_j \in \mathcal{T}$. From Proposition 2.15, calling $\{z_i\}_i$ a basis of $\mathcal{Z}^\circ$ and $\{y_i\}_i$ a basis of $\mathcal{Z}^{\circ\perp}$, the generators of $\mathcal{S}^*(T, A; \operatorname{Im} B)$ can only be of the following types:

1. $\sum_i \alpha_i z_i$, (from $\mathcal{B}_1$);
2. $\sum_i \alpha_i z_i + \sum_i \beta_i y_i$ (from $\mathcal{B}_2$),

where the coefficients α_i, β_i depend on the entries of A. Now note that the generators of $\mathcal{S}^*(T, A; \operatorname{Im} B)$ due to $\mathcal{B}_2$ cannot have only components in $\mathcal{Z}^\circ$ because the existence of the path $\mathcal{P}^{v_i, v_j}$ forces at least one α_i to be nonzero. As a consequence, when computing the intersection $\mathcal{Z}^\circ(A, \mathcal{B}; \mathcal{T}^\perp) \cap \mathcal{S}^*(T, A; \operatorname{Im} B)$ all the generators of $\mathcal{S}^*(T, A; \operatorname{Im} B)$ that are due to $\mathcal{B}_2$ cancel out. The remaining generators correspond to the controllable subspace of the pair $(A - BF_a, B_1)$. ∎

In the following example, the computation of $\mathcal{R}_{\mathcal{Z}^\circ}$ by means of Proposition 3.1 is shown.

Example 3.2

Let us consider the network depicted in Figure 3.4.
We note that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1 = \{v_1\} \subseteq \mathcal{Z}^\circ$ and $\mathcal{B}_2 = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A) = \{v_4\}$. We note that $F_a = \begin{bmatrix} 0 \\ e_4^\top A \end{bmatrix}$ is a friend of $\mathcal{Z}^\circ$. From Proposition 2.15, by taking an input matrix B with elementary columns s.t. $\operatorname{Im} B = \mathcal{B}$, the minimal input-containing (T, A)-invariant subspace is $\mathcal{S}^*(T, A; \operatorname{Im} B) = \operatorname{span}\{e_1, e_2, e_3, e_4, a_{54}e_5 + a_{64}e_6\}$. It is easy to see that the generators due to $\mathcal{B}_1$ are e_1, e_2 and e_3, while the generators due to $\mathcal{B}_2$ are e_4 and $a_{54}e_5 + a_{64}e_6$. We note that those last generators cannot lie entirely in $\mathcal{Z}^\circ$, thus they cancel out in the intersection with $\mathcal{Z}^\circ$. By Proposition 3.1, the maximal controllable subspace in $\mathcal{Z}^\circ$ corresponds to $\mathcal{R}_{\mathcal{Z}^\circ} = \mathcal{Z}^\circ(A, \mathcal{B}; \mathcal{T}^\perp) \cap \mathcal{S}^*(T, A; \operatorname{Im} B) = \operatorname{span}\{e_1, e_2, e_3\} = \sum_{i=1}^{n-1} (A - BF_a)^i \mathcal{B}_1$.

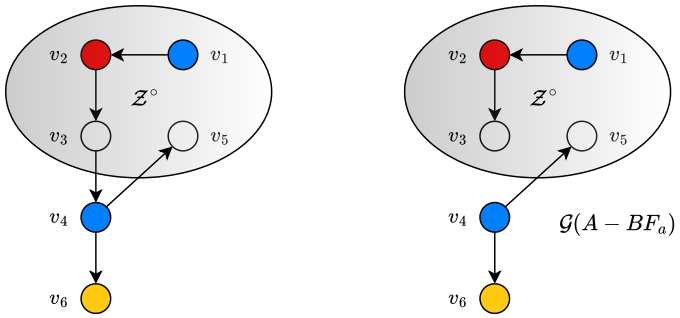


Figure 3.4: *Computation of the controllable subspace $\mathcal{R}_{\mathcal{Z}^\circ}$. On the right the network in closed loop is shown. Target nodes are in yellow, disturbances are in red, control inputs in blue.*

Remark 3.5. It is possible to associate to the subspace $\mathcal{R}_{\mathcal{Z}^\circ}$ of Definition 3.3, its corresponding node set $\mathcal{R}^\circ_{\mathcal{Z}^\circ}$. We call the minimal $\mathcal{B}_1$-containing $(A - BF_a)$-invariant node set in $\mathcal{Z}^\circ$, denoted by $\mathcal{R}^\circ_{\mathcal{Z}^\circ}$, the set of nodes defined by $\mathcal{R}^\circ_{\mathcal{Z}^\circ} = \mathcal{Z}^\circ(A, \mathcal{B}; \mathcal{T}^\perp) \cap \mathcal{S}^\circ(\mathcal{T}, A; \mathcal{B})$, where $\mathcal{S}^\circ(\mathcal{T}, A; \mathcal{B})$ is the minimal input-containing (T, A)-invariant node set, defined by the recursion of Proposition 2.15. It is possible to show that $\mathcal{R}^\circ_{\mathcal{Z}^\circ}$ contains all and only the nodes whose coordinates are nonzero in $\mathcal{R}_{\mathcal{Z}^\circ}$. Moreover, it is not difficult to prove that if $\mathcal{B}_1 = \emptyset$, i.e., if $\mathcal{B} = \mathcal{B}_2 = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A)$, then $\mathcal{B} \cap \mathcal{Z}^\circ = \emptyset$ and thus $\mathcal{R}^\circ_{\mathcal{Z}^\circ} = \emptyset$, which confirms that we need control inputs within $\mathcal{Z}^\circ$ to place the poles in $\mathcal{R}_{\mathcal{Z}^\circ}$.

The previous constructions are summarized in the following example.

Example 3.3

Let us consider the network depicted in Figure 3.5.

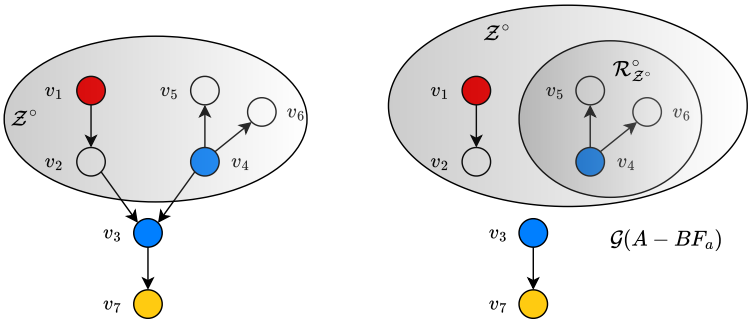


Figure 3.5: *Maximal controllable subspace in $\mathcal{Z}^\circ$. Target nodes are in yellow, disturbances are in red, control inputs in blue.*

We note that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1 = \{v_4\} \subseteq \mathcal{Z}^\circ$ and $\mathcal{B}_2 = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A) = \{v_3\}$, and that $F_a = \begin{bmatrix} 0 \\ e_3^\top A \end{bmatrix}$ is a friend of $\mathcal{Z}^\circ$. In closed loop, the minimal $\mathcal{B}_1$-containing

$(A - BF_a)$-invariant node set in $\mathcal{Z}^\circ$ is $\mathcal{R}^\circ_{\mathcal{Z}^\circ} = \{v_4, v_5, v_6\}$. We note that the maximal controllable subspace in $\mathcal{Z}^\circ$, $\mathcal{R}_{\mathcal{Z}^\circ}$, has generic dimension two, since it contains a dilation. The subspace $\mathcal{R}_{\mathcal{Z}^\circ}$ depends on the numerical entries in $A-BF_a$ but has nonzero components corresponding to the nodes in $\mathcal{R}^\circ_{\mathcal{Z}^\circ}$. As a side, note that F_p can externally stabilize $\mathcal{Z}^\circ$ without the need of additional self-loops, since in this case, $\mathcal{L} = \mathcal{Z}^{\circ\perp}$.

If $\mathcal{Z}^\circ$ is $(A-BF)$-invariant for some matrix F, then also $\mathcal{R}_{\mathcal{Z}^\circ}$ is $(A-BF)$-invariant. This result is summarized in the following Lemma.

Lemma 3.2. *If F_a is a friend of $\mathcal{Z}^\circ$, then F_a is a friend of $\mathcal{R}_{\mathcal{Z}^\circ}$.*

Proof: The statement is given in Property 4.1.7 of [10] and holds for all the self-bounded subspaces w.r.t. $\ker T$, so in particular for $\mathcal{R}_{\mathcal{Z}^\circ}$. ∎

W.r.t. the change of basis in equation (3.3), we can state the following Theorem about the internal stability of $\mathcal{Z}^\circ$.

Theorem 3.3. *If the pair $(A_{11}, \hat{B}_1)$ is stabilizable and $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{Z}^\circ}$, then $\exists\, F_1$ s.t. $\mathcal{Z}^\circ$ is internally stable.*

Proof: The statement is quite obvious since the disturbance excites only the controllable dynamics in the kernel of T, and the uncontrollable part in $\mathcal{Z}^\circ$ is stable by assumption. In fact, in the hypothesis of Theorem 3.3, if T_1 is a basis matrix of $\mathcal{R}_{\mathcal{Z}^\circ}$, i.e., $\operatorname{Im} T_1 = \mathcal{R}_{\mathcal{Z}^\circ}$, $[T_1\ T_2]$ is a basis matrix of $\mathcal{Z}^\circ$ and T_3 completes the basis in $\mathbb{R}^n$, by using the change of basis $T = [T_1\ T_2\ T_3]$ we get

$$
\begin{aligned}
T^{-1}(A - BF_a)T &= \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\ 0 & \hat{A}_{22} & \hat{A}_{23} \\ 0 & 0 & \hat{A}_{33} \end{bmatrix}, \\
T^{-1}B = T^{-1}[B_1\ B_2] &= \begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & 0 \\ 0 & \tilde{B}_2 \end{bmatrix}, \qquad (3.4) \\
T^{-1}D &= \begin{bmatrix} \hat{D}_1 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

We note that $\hat{A}_{21} = 0$ since F_a is a friend of $\mathcal{Z}^\circ$ and also of $\mathcal{R}_{\mathcal{Z}^\circ}$, by Lemma 3.2. Moreover, the structure of $T^{-1}D$ comes from the hypothesis of $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{Z}^\circ}$, and w.r.t. the change of basis in equations (3.3) and (3.4), we note that

$$
A_{11} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \quad A_{22} = \hat{A}_{33}.
$$

The hypothesis of (A_{11}, B_1) being stabilizable ensures that the uncontrollable eigenstructure, identified by the spectrum of $\hat{A}_{22}$, consists of only stable eigenvalues. We conclude that $\exists\, F_1$ s.t. $\mathcal{Z}^\circ$ is internally stable since the pair $(\hat{A}_{11}, \hat{B}_1)$ is controllable, and since $T^{-1}D$ has zero component on the uncontrollable dynamics. ∎

Remark 3.6 (Fixed internal eigenstructure of $\mathcal{Z}^\circ$). W.r.t. the change of basis in equation (3.4), we note that if the pair $(A_{11}, \hat{B}_1)$ is not stabilizable, some of the eigenvalues of $\hat{A}_{22}$ have positive real part. The eigenvalues of $\hat{A}_{22}$, representing the map $A - BF_a$ restricted to $\mathcal{Z}^\circ/\mathcal{R}_{\mathcal{Z}^\circ}$ are not accessible by $\begin{bmatrix} F_1^\top & 0 \end{bmatrix}^\top$, and are thus called the fixed internal eigenstructure of $\mathcal{Z}^\circ$, denoted by $\sigma^{\text{int}}_{\text{fixed}}(\mathcal{Z}^\circ)$. The associated eigenvectors correspond to the uncontrollable modes in $\ker T$, and $\sigma^{\text{int}}_{\text{fixed}}(\mathcal{Z}^\circ)$ are called the **invariant zeros** of (A, B, T) and the corresponding modes are called the **zero-dynamics** of (A, B, T). If an invariant zero is unstable, the system is called **non-minimum phase**, meaning that an initial condition in $\mathcal{Z}^\circ$ generates an unstable trajectory in $\mathcal{Z}^\circ$. This translates in $A - BF$ not being Hurwitz, while maintaining the disturbance invisible to the targets. For a detailed description of the system's zeros, connecting the invariant zeros with the transmission zeros in the cases of non-minimal realizations, we refer to [50, 51]. For more details on the fixed internal eigenstructure of $\mathcal{Z}^\circ$, we refer to [46].

If $\mathcal{D} \subseteq \mathcal{Z}^\circ$ but $\mathcal{D} \not\subseteq \mathcal{R}_{\mathcal{Z}^\circ}$ it means that the disturbance vector Dw entering in the state equation in the system (2.1) can excite the zero-dynamics of (A, B, T). Hence, while remaining invisible to the targets (hence undetectable from T), an attack that knows the value of the state at time t can place the poles in $\mathcal{Z}^\circ/\mathcal{R}_{\mathcal{Z}^\circ}$ and drive towards instability the internal dynamics of $\mathcal{Z}^\circ$. A disturbance that excites the zero-dynamics driving it towards instability is called a **zero-dynamics attack** (**ZDA**). If instead the disturbance consists of simple noise, the phase-minimality, guaranteed by the hypothesis of (A_{11}, B_1) being stabilizable, is sufficient to ensure the internal stability of $\mathcal{Z}^\circ$. In this case, we say that $\mathcal{D}$ is not a ZDA.

In the following, we give the complete characterization of the internal stability of $\mathcal{Z}^\circ$, in terms of the invariant zeros of the system.

Theorem 3.4. *If $(A_{11}, \hat{B}_1)$ is stabilizable and $\mathcal{D} \subseteq \mathcal{Z}^\circ$ but $\mathcal{D}$ is not a ZDA, then $\exists\, F_1$ s.t. $\mathcal{Z}^\circ$ is internally stable.*

Proof: Assume $\mathcal{D} \subseteq \mathcal{Z}^\circ$ and denote by $\mathcal{R}_{\mathcal{Z}^\circ}$ the maximal controllable subspace from $\mathcal{B}$ in $\mathcal{Z}^\circ$, and by $\mathcal{V}_{\mathcal{Z}^\circ}$ the maximal controllable subspace from $\mathcal{B} + \mathcal{D}$ in $\mathcal{Z}^\circ$. Since $\mathcal{D} \subseteq \mathcal{Z}^\circ$, $\mathcal{V}_{\mathcal{Z}^\circ}$ is defined as in Definition 3.3 as the controllable subspace of the pair $(A - BF_a, [B_1\ D])$, where B_1 is an input matrix with elementary columns s.t. $\operatorname{Im} B_1 = \mathcal{B}_1 = \mathcal{B} \cap \mathcal{Z}^\circ$ and D is the disturbance matrix with elementary columns s.t. $\operatorname{Im} D = \mathcal{D}$. Alternatively, as for $\mathcal{R}_{\mathcal{Z}^\circ}$ in Proposition 3.1, the subspace $\mathcal{V}_{\mathcal{Z}^\circ}$ can be computed by $\mathcal{Z}^\circ(A, \mathcal{B}; \mathcal{T}^\perp) \cap \mathcal{S}^*(T, A; \operatorname{Im}[B\ D])$, since $\mathcal{D} \subseteq \mathcal{Z}^\circ \implies \mathcal{Z}^\circ(A, \mathcal{B}; \mathcal{T}^\perp) = \mathcal{Z}^\circ(A, \mathcal{B} \cup \mathcal{D}; \mathcal{T}^\perp)$. Moreover, it holds that $\mathcal{V}_{\mathcal{Z}^\circ} \subseteq \mathcal{Z}^\circ$ and that $\operatorname{Im}[B\ D] \cap \mathcal{Z}^\circ \subseteq \mathcal{V}_{\mathcal{Z}^\circ}$ (and so $\operatorname{Im} B \cap \mathcal{Z}^\circ \subseteq \mathcal{V}_{\mathcal{Z}^\circ}$) meaning that $\mathcal{V}_{\mathcal{Z}^\circ}$ is self-bounded

w.r.t. $\ker T$ (the proof is similar to that of Theorem 3.2 by applying Lemma 3.1). As pointed out in the proof of Lemma 3.2, the friend F_a of $\mathcal{Z}^\circ$ is also a friend of $\mathcal{R}_{\mathcal{Z}^\circ}$ and of $\mathcal{V}_{\mathcal{Z}^\circ}$. Now, perform a change of coordinates $T = [T_1\ T_2\ T_3\ T_4]$ in $\mathbb{R}^n$ s.t. $\operatorname{Im} T_1 = \mathcal{R}_{\mathcal{Z}^\circ}$, $\operatorname{Im}[T_1\ T_2] = \mathcal{V}_{\mathcal{Z}^\circ}$, $\operatorname{Im}[T_1\ T_2\ T_3] = \mathcal{Z}^\circ$ and T_4 completes the basis in $\mathbb{R}^n$. We obtain

$$T^{-1}(A - BF_a)T = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\ 0 & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{33} & \bar{A}_{34} \\ 0 & 0 & 0 & \bar{A}_{44} \end{bmatrix},$$

$$T^{-1}B = T^{-1}[B_1\ B_2] = \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \bar{B}_2 \end{bmatrix}, \quad T^{-1}D = \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \\ 0 \\ 0 \end{bmatrix}. \tag{3.5}$$

The zero block structure is a consequence of F_a being a friend of $\mathcal{Z}^\circ$, $\mathcal{R}_{\mathcal{Z}^\circ}$ and $\mathcal{V}_{\mathcal{Z}^\circ}$. Note that the pair $(\bar{A}_{11}, \bar{B}_1)$ is controllable. W.r.t. the change of basis in equations (3.3) and (3.4), we note that

$$A_{11} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ 0 & \bar{A}_{22} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{33} \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} \bar{B}_1 \\ 0 \\ 0 \end{bmatrix}, \quad A_{22} = \bar{A}_{44}.$$

The eigenvalues of $\bar{A}_{22}$ and $\bar{A}_{33}$ are stable by the hypothesis of $(A_{11}, \hat{B}_1)$ being stabilizable. However, the pair $(\bar{A}_{22}, \bar{D}_2)$ is controllable from the disturbance, meaning that if $\mathcal{D}$ is a ZDA, then phase-minimality can break. ■

We conclude this section with an example of low dimension that clarifies Theorem 3.4.

Example 3.4

Let us consider the network depicted in Figure 3.6.

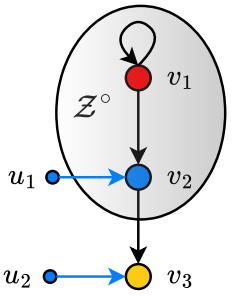


Figure 3.6: *Target nodes are in yellow, disturbances are in red, control inputs in blue. Note that v_3 is both a target and a control input.*

Assume $\mathcal{B}_1 = \{v_2\}$ and $\mathcal{B}_2 = \partial_+(\mathcal{Z}^\circ(\mathcal{B}), A) = \{v_3\}$, and consider the friend $F_a =$

$\begin{bmatrix} 0 \\ e_3^\top A \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix}$. It is easy to verify that $\mathcal{R}_{\mathcal{Z}^\circ} = \text{span}\{e_2\}$ and $\mathcal{V}_{\mathcal{Z}^\circ} = \text{span}\{e_1, e_2\} = \mathcal{Z}^\circ$. By applying the change of basis as in equation (3.5), we get $T = [e_2\ e_1\ e_3]$ s.t.

$$T^{-1}(A - BF_a)T = \begin{bmatrix} 0 & a_{21} & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ 0 & \bar{A}_{22} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{33} \end{bmatrix},$$

$$T^{-1}B = T^{-1}[B_1\ B_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & 0 \\ 0 & \bar{B}_3 \end{bmatrix},$$

$$T^{-1}D = \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \\ 0 \end{bmatrix}, \quad F_bT = \begin{bmatrix} \bar{F}_1 & 0 & 0 \\ 0 & 0 & \bar{F}_3 \end{bmatrix}.$$

We note that the pair $(\bar{A}_{11}, \bar{B}_1)$ is controllable, therefore placing λ_1 gives $\bar{F}_1 = a_{11} - \lambda_1$ ($\Re(\lambda_1) < 0$ for internal stability). Similarly, the pair $(\bar{A}_{33}, \bar{B}_3)$ is controllable, therefore placing λ_2 gives $\bar{F}_3 = -\lambda_2$ ($\Re(\lambda_2) < 0$ for external stability). Note that the stabilizability of the pair $(A_{11}, \hat{B}_1)$ as in Theorem 3.4 implies that $\bar{A}_{22}$ is stable, i.e., that the self-loop $\Re(a_{11}) < 0$. Moreover, we note that $\mathcal{D} \not\subseteq \mathcal{R}_{\mathcal{Z}^\circ}$, thus if the disturbance is a ZDA, phase-minimality can break by the pole placement of the pair $(\bar{A}_{22}, \bar{D}_2)$. If $\mathcal{D}$ is not a ZDA and $a_{11} < 0$, the formula for the stabilizing feedback friend solving the DDPSF-S is $F = F_a + F_b = \begin{bmatrix} F_1 \\ e_3^\top A + F_p \end{bmatrix} = \begin{bmatrix} 0 & a_{11} - \lambda_1 & 0 \\ 0 & a_{32} & -\lambda_2 \end{bmatrix}$. In Figure 3.7 the closed-loop system is depicted, while in Figure 3.8 some trajectories are simulated in the case of $a_{11} > 0$ (non-minimum phase) and $a_{11} < 0$ (minimum-phase), in the absence of disturbance. We note that the external stability of $\mathcal{Z}^\circ$ guarantees convergence on $\mathcal{Z}^\circ$ from any initial condition in $\mathbb{R}^n$.

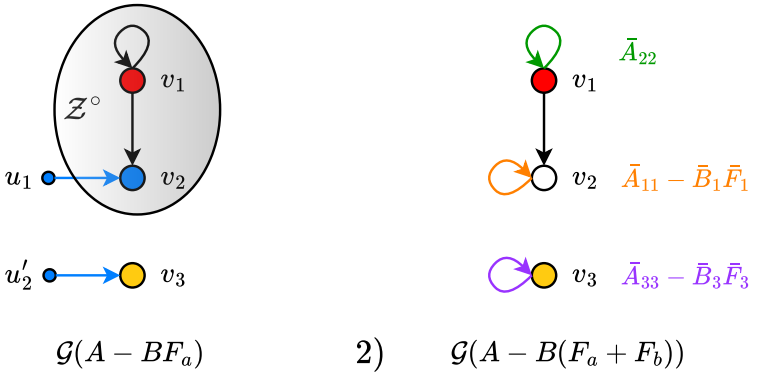


Figure 3.7: *Target nodes are in yellow, disturbances are in red, control inputs in blue. Note that v_3 is both a target and a control input.*

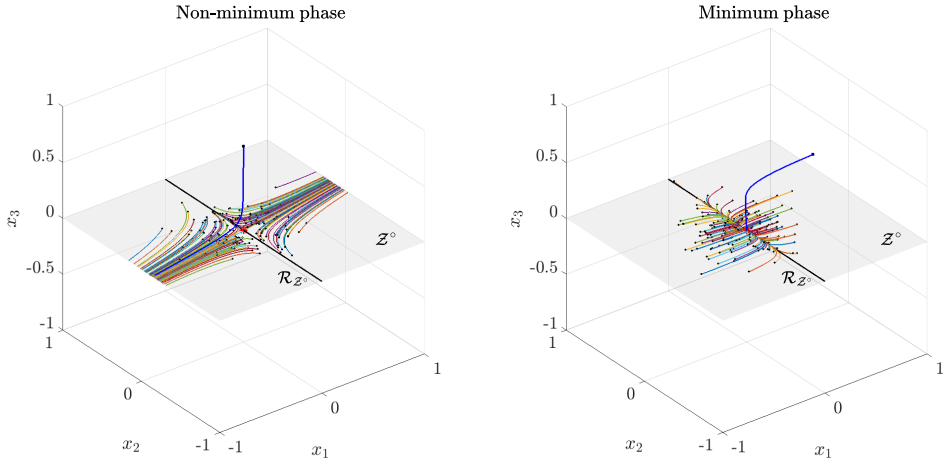


Figure 3.8: *External and internal stability of the $(A - BF)$-invariant node set $\mathcal{Z}^\circ$.*

4

Conclusion and future work

In this thesis, some core concepts of control theory, such as controllability, controlled invariance, and their duals, have been revisited and reinterpreted through the lens of networks. A significant effort has been devoted to collecting and unifying the relevant literature from both the geometric and structural perspectives, and to advancing a synthesis of these concepts into a coherent, albeit not yet complete, theory for networked systems.

Several important and novel relationships between the objects under investigation emerge as a result of this work. In particular, the links between the fundamental subspaces and their corresponding node sets are highlighted. Moreover, an effort has been made to investigate the differences between the node sets that inherit invariance properties and the node sets associated with fixed subspaces corresponding to the fundamental subspaces that appear in the structured-systems literature.

As a main contribution, this thesis demonstrates that control design strategies, particularly suited for large-scale systems, can be derived in terms of sets of nodes that inherit invariance properties. Specifically, the DDP is solved by blocking all paths from disturbances to target nodes by means of control input nodes (and measurement nodes in the case of the DDPOF and DDPDF). Such configurations of actuators and sensors can be identified in polynomial time using a modified version of the max-flow/min-cut algorithm, and they are optimal in the sense that they minimize the cardinality of the control input set (DDPSF) and of the measurement node set (DDPOF).

It is shown that these optimal configurations correspond exactly to the out-border of the maximal controlled invariant set of nodes $\mathcal{Z}^\circ$ and to the in-border of the minimal conditioned invariant set of nodes $\mathcal{S}^\circ$. As a consequence, explicit formu-

las for the associated feedback laws and dynamic compensators can be derived. Moreover, relatively to the DDPSF, these solutions are unique up to a free matrix parameter, which can be exploited to externally stabilize the invariant set of nodes $\mathcal{Z}^\circ$ in closed loop.

With regard to the DDPSF problem, it is shown that the optimal control input configuration (the configuration corresponding to the out-border of the maximal controlled invariant set of nodes $\mathcal{Z}^\circ$) cannot, in general, be used to internally stabilize $\mathcal{Z}^\circ$. For internal stabilization, additional control inputs within $\mathcal{Z}^\circ$ are required. The problem of externally and internally stabilizing an invariant set of nodes, after applying the decoupling friends in closed loop, can be addressed by introducing additional control inputs within $\mathcal{Z}^\circ$ and by adding self-loops of negative weight to specific nodes of the network. The allocation of these additional control inputs and the identification of the nodes that are critical for pole placement must be carried out so as to compensate for the presence of dilations and inaccessible nodes. The resulting conditions admit a purely graphical interpretation and build upon the structural notions developed throughout this thesis.

The notions and concepts presented in the background chapter are subsequently exploited in the papers. In order to avoid repeating the material already contained in those works, the background chapter focuses primarily on the underlying meaning of the definitions, rather than on the specific applications addressed in the papers. The original formulation in terms of node sets, however, originates from the work presented in Paper B. For this reason, in certain parts of the background chapter, some results and tools from Paper B have been incorporated in order to clarify theoretical questions concerning the relationships between node sets and their corresponding fundamental subspaces.

An excursus on the fixed controllable subspace, and in particular on its formal characterization in terms of matchings and the Dulmage-Mendelsohn decomposition, although not essential for understanding the papers, has been retained for completeness. The investigation of the interplay between the fundamental subspaces, their associated node sets, and the fixed subspaces has been indeed a central research theme throughout the development of this thesis.

I would like to conclude the discussion with an important remark. It is worth noting that the classical structural approach to the DDP, whose main contributors include Commault, Dion, and Van der Woude, leads to alternative graphical characterizations of the solvability conditions for the DDP compared to those obtained in our formulation. However, the manner in which these results have been derived follows a different line of reasoning with respect to the node-set formulation adopted in this thesis.

Indeed, as shown for instance in [37, 38], and in particular in [39], solvability conditions for the DDP are expressed in terms of infinite structural properties of the system transfer functions from control inputs, and from control inputs and disturbances to target outputs. These conditions admit a graphical interpretation in terms of graph linkings, that is, maximum sets of vertex-disjoint paths from the

control inputs to the targets and/or from the control inputs and disturbances to the targets.

The structured transfer matrix approach to the DDP encompasses both the disturbance decoupling problem with static feedback (DDPSF and DDPOF) and the dynamic feedback formulation (DDPDF), originally addressed in [34], and further developed in [52–54]. Related works have also employed the structure-at-infinity framework to tackle the non-interacting control problem, namely the problem of rendering the closed-loop transfer function diagonal by means of state feedback (see, e.g., [55]).

To this end, several algorithms have been proposed to compute the generic structure at infinity of a transfer function [40], see, e.g., [56–58]. These algorithms are used in the context of non-interacting control and, similarly to the minimal node cardinality disturbance decoupling problem addressed in this thesis, are based on variants of the max-flow/min-cut algorithm.

One of the main limitations of these approaches is that, although the solvability conditions for the DDP in structured systems are graphical and expressed in terms of graph linkings, they are not constructive. In contrast, the methods proposed in this thesis explicitly identify minimal sets of control input nodes and measurement nodes from which a feedback law achieving disturbance decoupling can be directly constructed.

For these reasons, and given that the underlying framework differs substantially from the one adopted in this work, this body of literature has not been explicitly included in the background chapter.

Summary of Contributions. The main contributions of this thesis can be summarized as follows:

- The reformulation of the classical notions of A-invariance, (A, B)-invariance and (C, A)-invariance in terms of sets of nodes and the graphical interpretation of their meaning. Firstly, by emphasizing the fundamental relationships between the geometric definitions of input-containing and output-nulling A-invariant, target-nulling (A, B)-invariant and disturbance-containing (C, A)-invariant subspaces, and their associated node sets. Secondly, by highlighting the difference between the input-containing and output-nulling A-invariant, target-nulling (A, B)-invariant and disturbance-containing (C, A)-invariant node sets, and the classical associated fixed subspaces that are generically contained and generically contain the input-containing and output-nulling A-invariant, target-nulling (A, B)-invariant and disturbance-containing (C, A)-invariant subspaces. (Chapter 2)
- The formal proof of the "fundamental advantage" of a temporal network. (Paper A)
- The partial characterization of the zero-measure set of realizations that do not increase controllability in a temporal network with fixed topology. (Paper A)

- A reinterpretation of controlled and conditioned A-invariance for sets of nodes. (Paper B)
- Sufficient graphical conditions for the solvability of DDPSF, DDPOF and DDPDF. (Paper B)
- The introduction of novel minimal node cardinality disturbance decoupling problems for state feedback, output feedback and dynamical feedback, and polynomial-time algorithms for finding the optimal solutions of the minimal node cardinality DDPSF and DDPDF. (Paper B)
- The derivation of closed-form formulas for the decoupling state feedback matrix, the output feedback matrix, and the dynamical compensator, in terms of the optimal solutions of the corresponding minimal node cardinality disturbance decoupling problems. (Paper B)
- An application of the graphical framework to the minimal node cardinality disturbance decoupling of coupled oscillators linearized at a stable synchronization state and a simulation of the results on the power network benchmark IEEE New England 39-bus system. (Paper C)
- An extension of the work of Paper B to account for disturbance decoupling with stability and pole placement over networks. (Chapter 3)

Future Work

Several directions emerge from this research:

- *Fixed controllable subspace:* Finding a comprehensive link between the eigenspaces of the zero eigenvalue and the structural Jordan canonical form, and the dilations' distribution for input-connected classes of networks.
- *Temporal network snapshots' degeneracy:* Fully characterize the zero-measure set of non-controllable temporal networks.
- *DDP with pole placement over networks:* Extending the disturbance decoupling notions over networks to handle stability and, more in general, pole placement. In particular, completing the work presented in Chapter 3 to account for DDPOF and DDPDF with stability.
- *Statistical validation of minimal node cardinality DDP solutions:* Simulating the optimal solutions and identifying the more suitable topologies for the DDP over networks on artificial networks (Erdös-Rény, Watts-Strogats, Scale-Free, etc.) and on real networks (from open source data repositories), with the different control law approaches. Investigate the quantities $\dim(\mathcal{Z}^*) - |\mathcal{Z}^\circ|$ and $|\mathcal{S}^\circ| - \dim(\mathcal{S}^*)$ in relation to the network topology.
- *Algorithm for minimal input/output cardinality DDPOF:* Finding, if possible, a polynomial-time algorithm for the solution of the minimal input/output cardinality DDPOF.

- *Nonlinear systems extension:* Extend the minimal node cardinality DDP to suitable classes of nonlinear systems, e.g., temporal networks/switching systems, compartmental models, etc.
- *Other geometric control problems over networks:* Investigate how other classical geometric control problems can be formulated and graphically reinterpreted when using the definitions of controlled and conditioned A-invariant node sets. For example, investigate how the problem of fault detection and isolation, and the entangled concepts of left-invertibility, translate when using the node set approach.

In conclusion, this thesis introduces a cohesive framework that streamlines the abstraction of classical geometric control theory, broadens its scope of application, and lays the groundwork for the formulation of new research directions.

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