Gaussian Mixture PHD Filtering with Variable Probability of Detection

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***Abstract*—The *probabilistic hypothesis density* (PHD) filter has grown in popularity during the last decade as a way to address the multi-target tracking problem. Several algorithms exist; for instance under linear-Gaussian assumptions, the *Gaussian mixture* PHD (GM-PHD) filter. This paper extends the GM-PHD filter to the common case with variable probability of detection throughout the tracking volume. This allows for more efficient utilization, *e.g.*, in situations with distance dependent probability of detection or occluded regions. The proposed method avoids previous algorithmic pitfalls that can result in a not well-defined PHD. The method is illustrated and compared to the standard GM-PHD in a simplified multi-target tracking example as well as in a realistic nonlinear underwater sonar simulation application, both demonstrating the effectiveness of the proposed method.**

I. Introduction

When dealing with multi-target tracking problems; the target identity can be of great importance or not. In the latter case, if the target identity is unimportant, the estimation problem can be formulated in such a way that it is possible to calculate the probability function for the number of targets in a given area. One such alternative to traditional association-based methods is based on *random finite sets* (RFS), [13]. In this formulation, the collection of individual targets is treated as a set-valued state, and the collection of individual measurements is regarded as set-valued observations. Using a Bayesian approach this method can handle multiple targets with association uncertainties in a cluttered environment, [7, 12, 20, 21]. This generalization of the single target Bayes' filter can be implemented using various methods, such as the *probability hypothesis density* (PHD) filter, [7, 12, 13], where the first order moment approximation is considered. For different PHD implementations see for instance, [11, 13, 15, 16, 18, 20–22]. The most commonly used approximation is probably the *Gaussian-mixture probably hypothesis density* (GM-PHD) filter [19]. It approximates the PHD using a Gaussian mixture in an efficient and easy to implement way. To allow for this, certain assumptions should be fulfilled; one of which is that the probability of detection p_D is constant. Approximations to allow for variable p_D have been suggested in literature, [13, 17], but not described and discussed in any detail. This paper suggests a method to allow for variable p_D in a GM-PHD, and discuss and exemplify the properties of the approximation in a way not seen in other papers.

The paper is organized as follows. In Sec. II the background theory for Bayesian estimation is presented together with an introduction to RFS. In Sec. III the PHD theory and the GM-PHD filter are introduced. Here the main contribution in the paper, extending the GM-PHD-theory to handle variable probability of detection is presented. In Sec. IV an illustrative example is given together with a realistic sonar application simulation extending the proposed method to a nonlinear observation as well. Finally, Sec. V concludes the paper.

II. Background

A. Target Tracking using Bayesian Filtering

In single target recursive Bayesian estimation the following time update and measurement update equations for the *probability density functions* (PDFs) need to be solved

$$p(x_{t+1}|y_{0:t}) = \int p(x_{t+1}|x_t)p(x_t|y_{0:t})\,dx_t \tag{1a}$$

$$p(x_t|y_{0:t}) = \frac{p(y_t|x_t)p(x_t|y_{0:t-1})}{\int p(y_t|x_t)p(x_t|y_{0:t-1})\,dx_t}, \tag{1b}$$

where x_t is the state vector at time t, and y_t is the observation at time t. The set of cumulative observations is denoted $y_{0:t}$. The PDFs are in general analytically intractable, but the *particle filter* (PF) [6, 8] provides a numerical solution incorporating both nonlinear and non-Gaussian systems. This extends the classic optimal filtering theory developed for linear and Gaussian systems, where the optimal solution is given by the *Kalman filter* [9, 10].

In many multi-target tracking applications, the target identity is of great importance. Hence, it is of interest to have an excellent method for observation to track association, particularly for dense target environments with cluttered observations. Traditional data association methods, [1–3, 14], based on the *global nearest neighbor* (GNN) method provides a simple data association method. For different clutter environments the *joint probabilistic data association* (JPDA), can be more attractive. The theoretically most accurate method is *multiple hypothesis tracking* (MHT) [14], but it is also the most computationally expensive method, since it is based on an extensive hypothesis tree, where all possible hypotheses are computed. To be able to implement this in practice, an efficient pruning algorithm is necessary. Usually these methods are based on the *(extended) Kalman filter* ((E)KF) or combinations of several such filters. For PF tracking, various approaches are also available.

In the sequel a method dealing with multi-target tracking without identity is presented in detail.

B. Muti-Target Tracking using Finite Set Statistics

In Sec. II-A the multi-target tracking problem was briefly discussed, assuming that target identity is important. If the target identity is unimportant this opens up for different solutions to the tracking problem. One such approach is to use finite sets statistics and ignore track labeling. Consider N targets with state-vectors $x^{(1)}, x^{(2)}, \ldots, x^{(N)}$. The problem can be divided in two categories, the first where the number of targets is known, but not the states. The second considers that the number of targets is unknown as well. The latter problem can be described using a *random finite set* (RFS) theory, [13]. It is possible to define a PDF for a RFS as

$$p(\{x^{(1)}, \ldots, x^{(N)}\}|N) = N!\, p(x^{(1)}, \ldots, x^{(N)}). \qquad (2)$$

In a similar way, now consider a RFS for the target states and measurements:

- X_t the target RFS with the set of all targets at time t.
- Y_t the RFS comprising measurements acquired at time t.
- $Y_{0:t}$ the measurement history RFS.

Inspired by the Bayesian single target solution in (1), the aim is to estimate $p(X_t|Y_{0:t})$. It can be shown, see *e.g.* [13], that the multi-target solution can be expressed similarly using set integrals,

$$p(X_{t+1}|Y_{0:t}) = \int p(X_{t+1}|X_t) p(X_t|Y_{0:t})\, \delta X_t, \qquad (3a)$$

$$p(X_t|Y_{0:t}) = \frac{p(Y_t|X_t) p(X_t|Y_{0:t-1})}{p(Y_t|Y_{0:t-1})}. \qquad (3b)$$

Note that these set-integrals are highly intractable. Hence, an approximation is needed. One approach is to reduce the problem and only estimate the first order moment.

III. Probabilistic Hypothesis Density Filter

To be able to approximate the problem as described in Sec. II-B, the expected mean for the set valued formulation must be defined using *finite set statistics* (FISST), [13]. However, the straightforward definition of expected value as $\int X p(X)\, \delta X$ is not applicable since addition of finite sets is not well-defined. A useful definition uses the expected mean of a function (converting sets to vectors). This leads to the *probability hypothesis density* (PHD) or intensity function

$$v(x) = \int \delta_X(x) p(X)\, \delta X. \qquad (4)$$

To get a realistic and useful multi-target tracking functionality using the PHD approach some model assumptions are usually made. Basically a motion model is defined as a transition likelihood. Existing targets can be updated using a survival probability and a spawning probability and the targets utilize an appearance model. The formulation also takes the probability of detection and false alarm rate into account. Exactly how these things are modeled depend on the actual implementation and which approximation of the PHD that is used. In the sequel this will be discussed using a GM-PHD approximation.

The PHD filter model assumptions and notations are summarized below:

- Motion model PDF: $p_{t|t-1}(x|\zeta)$.
- Survival probability for existing targets: $p_{S,t-1}(x_t)$.
- Spawning of new targets from existing ones: $\beta_{t|t-1}(x|\zeta)$.
- Appearance of new targets: $\gamma_t(x)$.
- Probability of detection: $p_D(x)$.
- False alarm model (Poisson distribution): $\kappa_t(y)$.
- Single target measurement likelihood: $p(y|x)$.

A. Gaussian-Mixture Probabilistic Hypothesis Density Filter

A naïve implementation of the PHD idea exhibits inherent exponential complexity. One method to get around this involves representing the underlying PHD with a Gaussian mixture, which can be efficiently handled. This leads to the popular GM-PHD filter, [13, 19]. As discussed in Sec. III, the PHD uses a motion model and an observation model. Also existing targets can be updated using a survival probability and a spawning probability etc. The GM-PHD filter formulation is based on several assumptions restricting the more general formulation discussed earlier, basically it requires a linear Gaussian motion and measurements, Gaussian birth and spawning processes, as well as constant probability of detection. In [5] uniform convergence for the GM-PHD was shown for EKF and *unscented Kalman filter* (UKF) implementations.

The GM-PHD filter model assumptions and notations are summarized below; before going in to a detailed description.

- Motion model $p_{t|t-1}(x|\zeta)$, linear and Gaussian,
  $$x = F_{t-1}\zeta + w_{t-1},$$
  where $w_{t-1} \sim \mathcal{N}(0, Q_{t-1})$.
- Survival probability for existing targets: $p_{S,t-1}$, constant.
- Spawning of new targets from existing ones: $\beta_{t|t-1}(x|\zeta)$, a Gaussian mixture as defined below.
- Appearance of new targets: $\gamma_t(x)$, a Gaussian mixture.
- Probability of detection: p_D, constant.
- False alarm model (Poisson distribution): $\kappa_t(y)$.
- Single target measurement likelihood: $p(y|x)$, linear and Gaussian,
  $$y = H_t x + e_t,$$
  where $e_t \sim \mathcal{N}(0, R_t)$.

The GM-PHD filter uses the following PHD representation

$$v_{t|t}(x) = \sum_{i=1}^{J_{t|t}} w_{t|t}^{(i)} \mathcal{N}(x; m_{t|t}^{(i)}, P_{t|t}^{(i)}), \qquad (5)$$

which uses $J_{t|t}$ weighted Gaussian components to represent the PHD. The Gaussian components are defined by $m_{t|t}$, $P_{t|t}$, and $w_{t|t}$; representing the mean, covariance, and weight, respectively. This representation is exact if the GM-PHD assumptions are fulfilled, otherwise only an approximation.

Given that the tracking problem can be modeled using constant $p_{D,t}$, a Gaussian-mixture birth process

$$\gamma_t(x) = \sum_{i=1}^{J_{\gamma,t}} w_{\gamma,t}^{(i)} \mathcal{N}(x; m_{\gamma,t}^{(i)}, P_{\gamma,t}^{(i)}) \tag{6a}$$

and a Gaussian-mixture spawning process

$$\beta_{t|t-1}(x|\zeta) = \sum_{i=1}^{J_{\beta,t}} w_{\beta,t} \mathcal{N}(x; F_{\beta,t-1}^{(i)}\zeta + d_{\beta,t-1}^{(i)}, Q_{\beta,t-1}^{(i)}), \tag{6b}$$

the PHD update are give by the following expressions.

Starting with the filtered PHD, $v_{t-1|t-1}$ in (5) the PHD time update is given by

$$v_{t|t-1}(x) = v_{S,t|t-1}(x) + v_{\beta,t|t-1}(x) + \gamma_t(x), \tag{7}$$

where $v_{S,t|t-1}(x)$ is the PHD of surviving target, $v_{\beta,t|t-1}(x)$ the PHD of new targets spawned from existing ones in this time update, and $\gamma_t(x)$ the PHD of newly born targets, as defined below.

The surviving target PHD is given by

$$v_{S,t|t-1}(x) = \sum_{i=1}^{J_{t-1|t-1}} w_{S,t}^{(i)} \mathcal{N}(x; m_{S,t}, P_{S,t}), \tag{8a}$$

where

$$w_{S,t}^{(i)} = p_{S,t-1}^{(i)} w_{t-1|t-1}^{(i)} \tag{8b}$$

$$m_{S,t}^{(i)} = F_{t-1} m_{t-1|t-1}^{(i)} \tag{8c}$$

$$P_{S,t}^{(i)} = F_{t-1} P_{t-1|t-1}^{(i)} F_{t-1}^T + Q_{t-1}, \tag{8d}$$

and $p_{S,t-1}$ is the probability that a target in time $t-1$ survives to time t.

The spawned target PHD is given by

$$v_{\beta,t|t-1}(x) = \sum_{i=1}^{J_{t-1|t-1}} \sum_{\ell=1}^{J_{\beta,t}} w_{\beta,t}^{(i,\ell)} \mathcal{N}(x; m_{\beta,t}^{(i,\ell)}, P_{\beta,t}^{(i,\ell)}) \tag{9a}$$

where

$$w_{\beta,t}^{(i,\ell)} = w_{t-1|t-1}^{(i)} w_{\beta,t}^{(\ell)} \tag{9b}$$

$$m_{\beta,t}^{(i,\ell)} = F_{\beta,t-1}^{(\ell)} m_{t-1|t-1}^{(i)} + d_{\beta,t-1}^{(\ell)} \tag{9c}$$

$$P_{\beta,t}^{(i,\ell)} = F_{\beta,t-1}^{(\ell)} P_{t-1|t-1}^{(i)} F_{\beta,t-1}^{(\ell)T} + Q_{\beta,t-1}^{(\ell)}. \tag{9d}$$

Together the equations (6)–(9) define how to predict the GM-PHD forward in time, and gives the time update step in the GM-PHD filter.

Provided the prediction PHD $v_{t|t-1}(x)$, given as a Gaussian mixture as described above, the PHD updated based on new measurements is

$$v_{t|t}(x) = (1 - p_{D,t}) v_{t|t-1}(x) + \sum_{y \in Y_t} v_{D,t}(x; y), \tag{10}$$

where the first term takes into account targets not detected (hence assumed to be where they were predicted to be), $p_{D,t}$ is the probability of detection, and the second term all the contributions from the observations. The contribution from the individual measurements are given by

$$v_{D,t}(x; y) = \sum_{i=1}^{J_{t|t-1}} w_{D,t}^{(i)}(y) \mathcal{N}(x; m_{D,t}^{(i)}(y), P_{D,t}^{(i)}) \tag{11a}$$

where

$$w_{D,t}^{(i)}(y) = \frac{p_{D,t} w_{t|t-1}^{(i)} q_t^{(i)}(y)}{\kappa_t(y) + p_{D,t} \sum_{\ell=1}^{J_{t|t-1}} w_{t|t-1}^{(\ell)} q_t^{(\ell)}(y)} \tag{11b}$$

$$m_{D,t}^{(i)}(y) = m_{t|t-1}^{(i)} + K_{D,t}^{(i)} \epsilon_t^{(i)}(y) \tag{11c}$$

$$P_{D,t}^{(i)} = (I - K_{D,t}^{(i)} H_t) P_{t|t-1}^{(i)} \tag{11d}$$

$$K_{D,t}^{(i)} = P_{t|t-1}^{(i)} H_t^T (S_t^{(i)})^{-1} \tag{11e}$$

$$\epsilon_t^{(i)}(y) = y - H_t m_{t|t-1}^{(i)} \tag{11f}$$

$$S_t^{(i)} = H_t P_{t|t-1}^{(i)} H_t^T + R_t \tag{11g}$$

$$q_t^{(i)}(y) = \mathcal{N}(y; \epsilon_t^{(i)}, S_t^{(i)}), \tag{11h}$$

which concludes how to infer the information from new measurements into the predicted GM-PHD.

Repeatedly applying (7) and (10), under the given assumptions, yields an exact representation of how the PHD progresses over time.

However, this solution suffers from exponential complexity growth, as the number of components used in the GM-PHD representation increases combinatorially in each step. The solution to this problem is to, regularly, reduce the representation to keep the number of components down, yielding an approximative but manageable PHD representation. With the reduction step in place, the GM-PHD filter becomes a feasible alternative to other multi-target tracking algorithms. See [19] for one way to perform this GM-PHD reduction.

B. Extensions to GM-PHD Filtering

It was early recognized that the GM-PHD filter could be extended to nonlinear system dynamics and/or measurement models. The suggested solution is to observe that each Gaussian component in the filter is updated using a separate/independent Kalman filter. In [19] the authors' suggest using linearized dynamics and measurement equations, resulting in EKF like updates or to utilize the unscented transform, resulting in a UKF inspired solution.

Another extension, also mentioned in [19], is the possibility to use components with negative weights in order to, for instance, better shape the birth or spawning processes. This comes with the draw back that, during the necessary PHD reduction step, the PHD might actually end up being negative in some areas, causing the GM-PHD to break down. Hence, great care must in those cases be used when implementing the reduction step to ensure this does not happen. This is nontrivial, and no generally accepted algorithm is available.

With similar arguments it is easy to show that the probability of detection cannot be modeled using a Gaussian (mixture), as Gaussianity is not preserved throughout the computations;

hence leaving the assumption of a constant p_D for the GM-PHD filter. This has previously been suggested as part of other GM-PHD adaptions [17], but not been treated in sufficient detail on its own in literature.

C. GM-PHD Filter for State Dependent p_D

In this paper a different method to handle state dependent probability of detection is proposed. By approximating $p_D(x)$ with a constant around the center point of the currently considered Gaussian component, the problems with the extensions described in Sec. III-B can be avoided. At the same time, this extension to the GM-PHD formulation, which in many cases is restrictive, is able to handle, *e.g.*, cases where $p_D(x)$ depends on the distance between the sensor and the target and cases where there are blind spots with respect to the sensor in the tracking volume. As described below, the algorithmic changes needed to facilitate this approximation are minor.

The suggested approximation affects the computation of both the terms in (10). In the first term, p_D is computed independently for each component, *i.e.*,

$$(1-p_{D,t})v_{t|t-1}(x) = \sum_{i=1}^{J_{t|t-1}} (1-p_{D,t}) w^{(i)}_{t|t-1} \mathcal{N}(x; m^{(i)}_{t|t-1}, P^{(i)}_{t|t-1}) \quad (12a)$$

becomes

$$\sum_{i=1}^{J_{t|t-1}} \big(1-p_{D,t}(m^{(i)}_{t|t-1})\big) w^{(i)}_{t|t-1} \mathcal{N}(x; m^{(i)}_{t|t-1}, P^{(i)}_{t|t-1}). \quad (12b)$$

In second term (11a), the way that the weights (11b) are computed must be modified. Straightforward application of the approximation yields

$$w^{(i)}_{D,t}(y) = \frac{p_{D,t}(m^{(i)}_{t|t-1}) w^{(i)}_{t|t-1} q^{(i)}_t(y)}{\kappa_t(y) + p_{D,t}(m^{(i)}_{t|t-1}) \sum_{\ell=1}^{J_{t|t-1}} w^{(\ell)}_{t|t-1} q^{(\ell)}_t(y)}. \quad (13)$$

No other changes to the algorithm are needed, making the extension very straightforward and easy to implement.

A few comments and remarks on the proposed extension:

- First of all, it should be noted that for the special case that $p_D(x)$ is constant, the described GM-PHD filter extension reduces to the standard GM-PHD filter algorithm, as should be expected in this case.
- The described approximation is valid when the probability of detection, $p_D(x)$ can be assumed to vary slowly compared to the Gaussian components in v_t. That is when

$$\int p_D(x) \mathcal{N}(x; m, P)\, dx \approx \int p_D(m) \mathcal{N}(x; m, P)\, dx, \quad (14)$$

  the approximation can be expected to be good. This can be expected to be the case when the variations of $p_D(x)$ are small around m on the support given by the Gaussian, which is determined by P.
- It was previously mentioned that using negative weights may result in a infeasible PHD after the complexity reducing step. This can happen even if the GM-PHD is correct before the reduction step, and is a result of how Gaussian modes are selected and merged to reduce the components of the GM-PHD. The suggested method does not suffer the risk of producing invalid GM-PHD representations. All weights maintain nonnegative values throughout the algorithm (including the important reduction), as can easily be verified given that $0 \leq p_D(x) \leq 1$ by definition, and hence $0 \leq 1 - p_D(x) \leq 1$.
- The changes to the GM-PHD algorithm are limited to just a few equations, which servers beneficial when trying to implement the algorithm. It should be possible to implement the extended GM-PHD almost as efficiently as the regular GM-PHD.

IV. Simulations

In this section, the suggested extension to the GM-PHD filter algorithm will be illustrated and compared to the standard GM-PHD filter using two simulations. The first example is artificial to clearly illustrate the behavior, whereas the second one is more closely resembling an actual sonar tracking scenario. For details on sonar tracking using PHD-filtering, see for instance [4].

A. Illustrative Multiple Target Tracking Example

In the first simulation, assume a quadratic tracking volume with $p_D = 95\%$ except for a circular region in the center, where instead $p_D = 5\%$ with a steep transition in between. (This could be the case when tracking ground targets and there is a hole in the ground occluding the targets.) Further, assume three different targets (T_1, T_2, and T_3) and that targets enter the tracking volume from one of the corners; as depicted in Fig. 1.

The setup details are given below. Assume the target state consist of position and velocity, $x = (\mathsf{x}, \mathsf{y}, v_\mathsf{x}, v_\mathsf{y})^T$, $R = \sqrt{\mathsf{x}^2 + \mathsf{y}^2}$ and

$$\gamma(x) = \sum_{\substack{x\in\{-45,45\}\\ y\in\{-45,45\}}} 0.05\mathcal{N}\big(x; (\mathsf{x}\ \mathsf{y}\ 0\ 0)^T, 5^2 I_4\big) \quad (15a)$$

$$p_D(x) = \begin{cases} 0.05 & \text{if } R \leq 15, \\ 0.05 + 0.18(R-15) & \text{if } 15 < R < 20, \\ 0.95 & \text{otherwise,} \end{cases} \quad (15b)$$

(which is approximated with $p_D = 0.95$ in the regular GM-PHD filter), and

$$x_{t+1} = \begin{pmatrix} I_2 & TI_2 \\ 0 & I_2 \end{pmatrix} x_t + \begin{pmatrix} TI_2 \\ I_2 \end{pmatrix} w_t \quad (15c)$$

$$y_t = \begin{pmatrix} I_2 & 0_2 \end{pmatrix} x_t + e_t, \quad (15d)$$

where $w_t \sim \mathcal{N}(0, I_2)$ and $e_t \sim \mathcal{N}(0, I_2)$ are white and mutually independent noise, $T = 0.5$, and the number of false detections is $\mathcal{P}o(5)$ distributed and spatially uniformly distributed in the full tracking volume $-50 \leq \{\mathsf{x}, \mathsf{y}\} \leq 50$.

A standard GM-PHD and the GM-PHD with the variable p_D extension have been applied to data generated according to

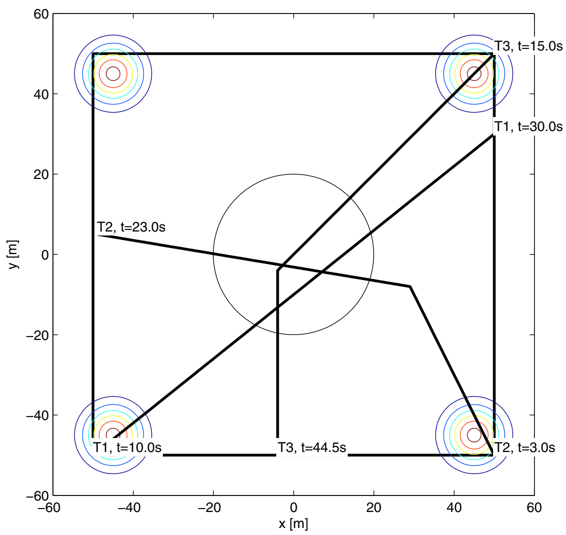

Fig. 1. Illustration of the simulated tracking scenario. The circles in the corners indicate birth regions, whereas the circle in the middle indicates the region with decreased probability of detection. The target trajectories are indicated by T_1–T_3 where start and end times of trajectories are also given.

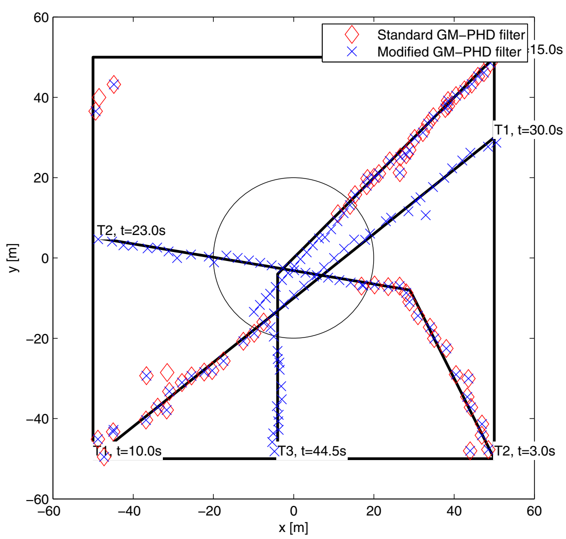

Fig. 2. Illustration of the simulated tracking scenario. The marker indicate the major Gauss modes in the respective estimated PHD. As many modes are given from each time instance as the rounded total weight of the PHD.

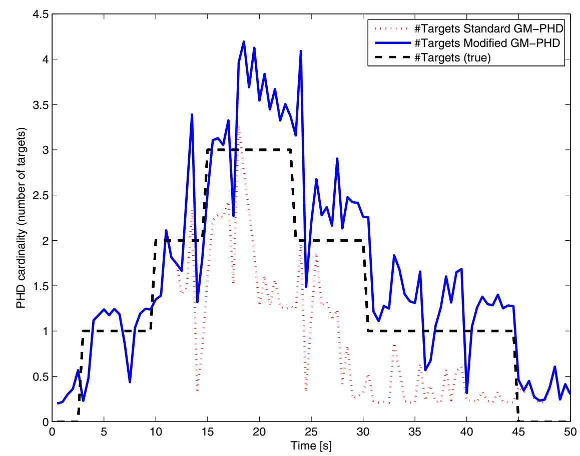

Fig. 3. Estimated number of targets for the standard and modified GM-PHD compared to the true number. As seen, for those regions where the probability of detection is quite different from the constant assumption the standard method performs poorly compared to the modified version.

the above description. The result is given in Fig. 2 where it is clear that the standard GM-PHD formulation cannot handle the zone with lower p_D in the middle of the tracking volume, the target mass is lost almost immediately as the number of detections drop below the given p_D. This is expected given that the PHD filter is known to react fast to changes in the cardinality. The modified GM-PHD on the other hand is able to able to correctly handle the changed p_D and the track remains throughout the passage through the center. Also, as expected, the filter does not capture the change of direction that T_3 performs in the occluded zone; and the estimate continues as if there were no change in direction. However the process noise is big enough to capture the change once the target becomes observable again.

A similar picture is conveyed by Fig. 3, which shows the cardinality of the two GM-PHD filters together with the true number of target in each instance. Overall the modified GM-PHD estimate is better than the standard form, which obviously drops the targets more or less directly when the enter the dead zone in the center. Overall the cardinality could have been better estimated using a cardinalized GM-PHD; however, the principal behavior would be the same. The variable p_D extension suggested here easily carries over to the cardinalized GM-PHD.

B. Sonar Target Tracking

In the second simulation, a sonar application inspired by an authentic dataset is described and evaluated with three targets. The propagation of high frequency acoustic waves, so-called sonar, in shallow waters or in environments with a lot of obstacles can be quite complex. In this example the focus is on the probability of detection. It can be modeled as being roughly inversely proportional to the range in cube. Both the proposed state-dependent $p_D(x)$ extension to the GM-PHD filter and the normal GM-PHD filter has been used to track targets in noisy measurements. It is worth noting that the measurement equation in this case is nonlinear, and that the extended Kalman filter version of the GM-PHD filter is used. This shows that the suggested extension to the GM-PHD filter also works with this approximation of the filter.

The setup details are given below. Assume the target state consist of position and velocity, $x = (\mathsf{x}, \mathsf{y}, v_\mathsf{x}, v_\mathsf{y})^T$, and that the range from the origin where the sensor is located is $R = \sqrt{\mathsf{x}^2 + \mathsf{y}^2}$ and the bearing $\psi = \arctan(\mathsf{y}/\mathsf{x})$. The probability of detection

$$p_D(x) = 0.9 - 4 \cdot 10^{-7} R^2 - 1.6 \cdot 10^{-9} R^3, \tag{16a}$$

when $|\psi| < 25^\circ$ and $0 \leq R \leq 750$, and otherwise $p_D(x) = 0$,

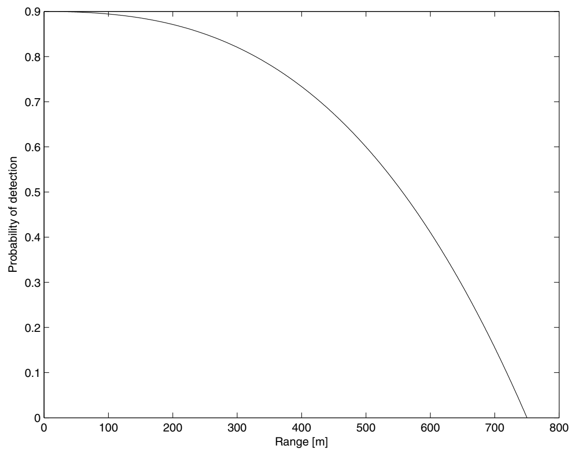

Fig. 4. The probability of detection as a function of distance, assuming that the horizontal aspect angle is $\pm 25°$.

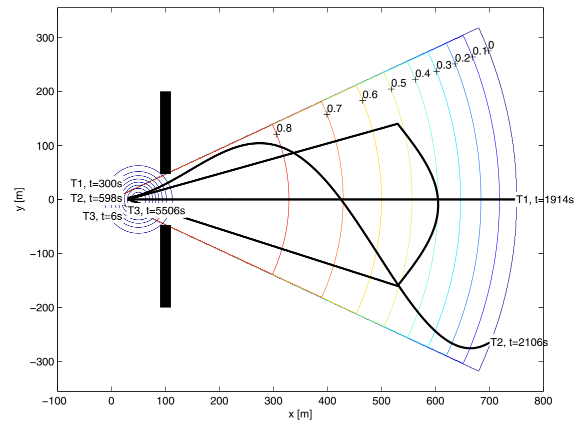

Fig. 5. The figure depicts the sonar application scenario, where the GM-PHD is applied to a range dependent probability of detection as visualized by the contour plots, an the circles to the left the birth region. The dark vertical lines represent pier that physically limit the field of view.

as depicted in Fig. 4. For the regular GM-PHD $p_D = 0.7$ is used as a compromise to maintain good tracking as the range increases, while at the same time have acceptable behavior at short range. Furthermore the birth process, the target dynamics and measurement equation are

$$\gamma_t(x) = 0.01\mathcal{N}\big(x; (50, 0, 0, 0)^T, \text{diag}(30^2 \cdot I_2, I_2)\big) \qquad \text{(16b)}$$

$$x_{t+1} = \begin{pmatrix} I_2 & TI_2 \\ 0 & I_2 \end{pmatrix} x_t + \begin{pmatrix} TI_2 \\ I_2 \end{pmatrix} w_t \qquad \text{(16c)}$$

$$y_t = h(x_t) + e_t = \begin{pmatrix} R \\ \psi \end{pmatrix} + e_t, \qquad \text{(16d)}$$

where $w_t \sim \mathcal{N}(0, I_2)$ and $e_t \sim \mathcal{N}\big(0, \text{diag}((1°)^2, 2^2)\big)$ are white and mutually independent noise, $T = 2$, and the number of false detections is $\mathcal{P}o(20)$ distributed and spatially uniformly distributed in the full measurement volume.

The presented underwater scenario describes one diver swimming from the left to the right, where the detection probability decrease with distance. A second diver zig-zag through the tracking volume, and the third one first heads away from the sonar, before returning back to where he came from. This is illustrated in Fig. 5. Fig. 6 shows the tracking result using the two different filters. Again, the filter with the incorrect p_D description lose track of the targets as they enter regions with lower than modeled probability of detection. It is also clear that the tracking is more difficult in areas with low p_D than in areas with high p_D, even when correctly modeled.

In Fig. 7 number of estimated targets with the two filters are given. In this case the difference is less prominent, but overall the modified GM-PHD filter still performs better. For high probability of detection regions the methods perform similarly. However, the standard method with constant p_D performs badly when the detection rate goes down. It is also possible to see the effects of underestimating the probability of detection at short range, which leads to overestimation of the number of targets there.

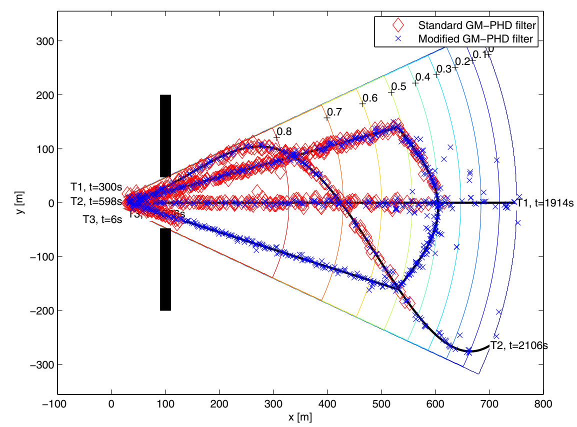

Fig. 6. A comparison between the tracks from the two methods. As seen the classical method based on constant p_D has a track loss, whereas the proposed method can handle the detection probability variability quite well. To get better visualization the data points are decimated before plotting.

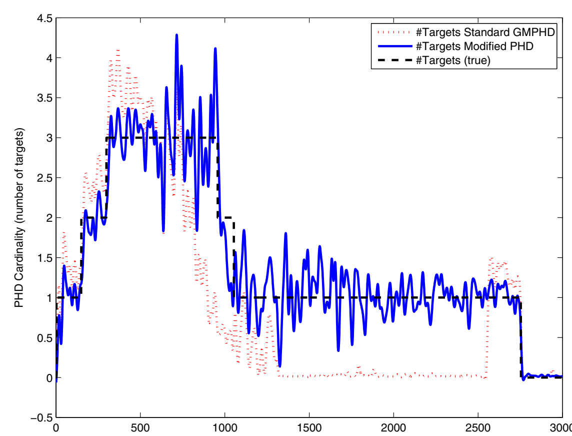

Fig. 7. Low pass filtered cardinality (estimated number of targets) for the standard and modified GM-PHD compared to the true number. As seen the standard method and the modified yield similar results in the beginning, when the probability of detection is high. At the end, the modified method is superior since the standard one lost the track completely.

V. Conclusion

In this paper, an extension to the standard *Gaussian-mixture probability hypothesis density* (GM-PHD) filter that allows for varying detection probabilities, p_D, has been derived. The method relies on approximating $p_D(x)$ as constant around the centers of each Gaussian component of the PHD as they are handled, and can be combined with other extensions of the GM-PHD. The approximation is valid when the variation of $p_D(x)$ is small compared to the uncertainty of the involved Gaussian components.

The effectiveness of the algorithm is shown in two simulation studies; one highly simplified, another more realistic sonar simulation extending the problem for nonlinear observations. The promising results indicate that the method has good potential also in more realistic settings, *e.g.*, where the probability decreases with the distance between the target and the sensor, or is affected by occlusions.

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